Global Strong Solution to a Nonlinear Dispersive Wave Equation

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Abstract

In this paper, compared with the previous results, a new global existence for strong solutions to the equation is acquired provided that the potential $(1 - \partial_x^2)u_0$ changes sign on $\mathbf{R}$, which improves considerably the previous result.

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1 Introduction

In this article, we will consider the nonlinear dispersive wave equation

$$u_t - u_{txx} + (a + b)uu_x - au_xu_{xx} - buu_{xxx} + \lambda(u - u_{xx}) = 0, \quad (1)$$

where $a > 0$, $b > 0$ and $\lambda \geq 0$ are arbitrary constants, u is the fluid velocity in the x direction, $\lambda(u - u_{xx})$ represents the weakly dissipative term.

For $a = 2$, $b = 1$ and $\lambda = 0$ in Eq.(1), Eq.(1) becomes the famous Camassa-Holm equation [1, 2, 3]

$$u_t - u_{txx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0, \quad t > 0, \quad x \in \mathbf{R}, \quad (2)$$

where $u(x, t)$ represents the free surface above a flat bottom. As a model to describe the shallow water motion, Eq.(2) has a bi-Hamiltonian structure and infinite conservation laws and is completely integrable.

For $a = 2$, $b = 1$ and $\lambda > 0$ in Eq.(1), Eq.(1) changes into weakly dissipative Camassa-Holm equation [4]

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \lambda(u - u_{xx}) = 0, \quad t > 0, \quad x \in \mathbf{R}. \quad (3)$$

For $a = 3$, $b = 1$ and $\lambda = 0$ in Eq.(1), Eq.(1) becomes the classical Degasperis-Procesi equation[5, 6]

$$u_t - u_{txx} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0, \quad t > 0, \quad x \in \mathbf{R}. \quad (4)$$

As a model to describe the shallow water wave, Degasperis-Procesi equation (4) also satisfies complete integrability, bi-Hamiltonian structure, infinite conservation laws and peakon soliton solutions which are analogous to the Camassa-Holm. However, they are quite different model(see [5]).

For $a = 3$, $b = 1$ and $\lambda > 0$ in Eq.(1), Eq.(1) is read as weakly dissipative Degasperis-Procesi equation[7]

$$u_t - u_{txx} + 4uu_x - 3u_x u_{xx} - uu_{xxx} + \lambda(u - u_{xx}) = 0, \quad t > 0, \quad x \in \mathbf{R}. \quad (5)$$

In [8], Lai and Wu study global existence and blow-up to Eq.(1) with $\lambda = 0$ assumed that the potential $(1 - \partial_x^2)u_0$ does not change sign on $\mathbf{R}$. To our best knowledge, the global strong solution of Eq.(1) under the condition $y_0 \leq 0$ for $x \leq x_0$ and $y_0 \geq 0$ for $x \geq x_0$ seems not have been investigated. Present paper is mainly concerned with global strong solution to Eq.(1) provided that the potential $(1 - \partial_x^2)u_0$ changes sign on $\mathbf{R}$. Since Eq.(1) is a generalization of Camassa-Holm equation and Degasperis-Procesi equation, Eq.(1) loses some important conservation laws that they possess. In the paper, we mainly depend on some useful prior estimates from the equation and the good method presented in Liu and Yin [6] to obtain the blow-up solution for the equation.

2 Preliminary Notes

We denote by $*$ the convolution. Note that if $G(x) := \frac{1}{2}e^{-|x|}$, $x \in \mathbf{R}$, then $(1 - \partial_x^2)^{-1}f = G * f$ for all $f \in L^2(\mathbf{R})$ and $G * (u - u_{xx}) = u$. Using this identity, the Cauchy problem of Eq.(1) becomes

$$\begin{cases} u_t + buu_x + \partial_x G * [\frac{a}{2}u^2 + \frac{3b-a}{2}(u_x)^2] + \lambda u = 0, & t > 0, \quad x \in \mathbf{R}, \\ u(0, x) = u_0(x), \end{cases} \quad (6)$$

which is equivalent to

$$\begin{cases} y_t + byy_x + ayu_x + \lambda y = 0, & t > 0, \quad x \in \mathbf{R}, \\ y = u - u_{xx}, \\ u(0, x) = u_0(x). \end{cases} \quad (7)$$

We firstly establish the local well-posedness of solution and blow-up scenario for problem (6).

Lemma 2.1. [9] *Given $u_0 \in H^s(s > \frac{3}{2})$, there exist a maximal $T = T(u_0)$ and a unique solution u to problem (6), such that*

$$u = u(\cdot, u_0) \in C\left([0, T]; H^s(\mathbf{R})\right) \cap C^1\left([0, T]; H^{s-1}(\mathbf{R})\right).$$

Lemma 2.2. [9] *Let $u_0 \in H^s, s \geq 2$, and u be the corresponding solution to problem (6) with time T . Then $T < \infty$ if and only if*

$$\liminf_{t \rightarrow T} \left\{ \inf_{x \in \mathbf{R}} \left[u_x(t, x) \right] \right\} = -\infty.$$

3 Global strong solution

In this section, we discuss the global existence of solution to problem (6).

Lemma 3.1. *Let $u_0 \in H^s(\mathbf{R})(s > 3)$, and T be the maximal existence time of the corresponding solution u to Eq.(1). Then system (18) has a unique solution $q \in C^1([0, T) \times \mathbf{R}; \mathbf{R})$. Moreover, the map $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbf{R}$ with*

$$q_x(t, x) = \exp \left(\int_0^t b u_x(s, q(s, x)) ds \right) > 0, \quad \forall (t, x) \in [0, T) \times \mathbf{R}$$

and

$$y(t, q(t, x)) q_x^2(t, x) = y_0(x) \exp \left(\int_0^t [-(a - 2b) u_x(s, q(s, x)) - \lambda] ds \right).$$

Proof. From Theorem 2.1, we have $u \in C^1([0, T); H^{s-1}(\mathbf{R}))$ and $H^{s-1} \in C^1(\mathbf{R})$. We conclude that both functions $u(t, x)$ and $u_x(t, x)$ are bounded, Lipschitz in space and C^1 in time. Applying the existence and uniqueness theorem of ordinary differential equations implies that system (18) has a unique solution $q \in C^1([0, T) \times \mathbf{R}, \mathbf{R})$.

Differentiating (18) with respect to x leads to

$$\begin{cases} \frac{d}{dt} q_x = b u_x(t, q) q_x, & t \in [0, T), \quad b > 0, \\ q_x(0, x) = 1, & x \in \mathbf{R}, \end{cases} \quad (8)$$

which yields

$$q_x = \exp \left(\int_0^t b u_x(s, q(s, x)) ds \right). \quad (9)$$

For every $T' < T$, using the Sobolev embedding theorem gives rise to

$$\sup_{(s,x) \in [0,T') \times R} |u_x(s,x)| < \infty.$$

It is inferred that there exists a constant $K_0 > 0$ such that $q_x \geq e^{-K_0 t}$ for $(t,x) \in [0,T) \times R$.

By computing directly, we derive

$$\frac{d}{dt}[y(t, q(t, x))q_x^2(t, x)] = -(a - 2b)u_x(t, x)yq_x^2 - \lambda yq_x^2, \quad (10)$$

which results in

$$y(t, q(t, x))q_x^2(t, x) = y_0(x) \exp \left(\int_0^t [-(a - 2b)u_x(s, q(s, x)) - \lambda] ds \right).$$

The proof of Lemma 3.1 is completed.

Lemma 3.2. *Provided that $u_0 \in H^s(\mathbf{R}) \cap L^1(\mathbf{R})$, $s \geq \frac{3}{2}$, if $y_0 = u_0 - u_{0xx}$ satisfies*

$y_0 \leq 0$ for $x \leq x_0$, $y_0 \geq 0$ for $x \geq x_0$ and $y(t, x) = u(t, x) - u_{xx}(t, x)$, then, for $t \in \mathbf{R}_+$, it holds

$$(i) \|u\|_{L^1} \leq e^{-\lambda t} \|u_0\|_{L^1} \text{ on } \mathbf{R},$$

$$(ii) \|u\|_{H^1}^2 \leq \|u_0\|_{H^1}^2 \exp \left(-2\lambda t + \frac{|a-2b|}{\lambda} \|u_0\|_{L^1} (1 - e^{-\lambda t}) \right).$$

Proof. Since $q(t, x)$ is an increasing diffeomorphism of R with $q_x(t, x) > 0$ with respect to time t . We deduce from the assumption that for $t \in [0, T)$,

$$\begin{cases} y(t, x) \leq 0, & \text{if } x \leq q(t, x_0), \\ y(t, x) \geq 0, & \text{if } x \geq q(t, x_0), \end{cases} \quad (11)$$

and $y(t, q(t, x_0)) = 0$.

Integrating the first equation of problem (6) with respect to x in interval $(-\infty, q(t, x_0)]$ yields

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{q(t, x_0)} u dx &= -b \int_{-\infty}^{q(t, x_0)} u u_x dx - \int_{-\infty}^{q(t, x_0)} \partial_x (G * [\frac{a}{2} u^2 + \frac{3b-a}{2} (u_x)^2]) dx \\ &\quad - \lambda \int_{-\infty}^{q(t, x_0)} u dx. \end{aligned} \quad (12)$$

On the other hand, integrating the first equation of problem (6) with respect to x in interval $[q(t, x_0), +\infty)$ leads to

$$\begin{aligned} \frac{d}{dt} \int_{q(t, x_0)}^{\infty} u dx &= -b \int_{q(t, x_0)}^{\infty} u u_x dx - \int_{q(t, x_0)}^{\infty} \partial_x (G * [\frac{a}{2} u^2 + \frac{3b-a}{2} (u_x)^2]) dx \\ &\quad - \lambda \int_{q(t, x_0)}^{\infty} u dx. \end{aligned} \quad (13)$$

Subtracting (12) from (13), one has

$$\begin{aligned} \frac{d}{dt} \left(\int_{q(t,x_0)}^{\infty} u dx - \int_{-\infty}^{q(t,x_0)} u dx \right) &= 2G * \left[\frac{a}{2} u^2 + \frac{3b-a}{2} (u_x)^2 \right] \Big|_{x=q(t,x_0)} \\ &\quad - b u^2 \Big|_{x=q(t,x_0)} - \lambda \left(\int_{q(t,x_0)}^{\infty} u dx - \int_{-\infty}^{q(t,x_0)} u dx \right), \end{aligned}$$

which results in

$$\frac{d}{dt} \int_{-\infty}^{\infty} |u| dx \leq -\lambda \int_{-\infty}^{\infty} |u| dx.$$

Hence, we get

$$\|u\|_{L^1} \leq e^{-\lambda t} \|u_0\|_{L^1}.$$

This proves (i).

Using $u = G * y$, one has

$$u(t, x) = \frac{e^{-x}}{2} \int_{-\infty}^x e^{\xi} y(\xi) d\xi + \frac{e^x}{2} \int_x^{\infty} e^{-\xi} y(\xi) d\xi, \quad (14)$$

from which we obtain

$$u_x(t, x) = -\frac{e^{-x}}{2} \int_{-\infty}^x e^{\xi} y(\xi) d\xi + \frac{e^x}{2} \int_x^{\infty} e^{-\xi} y(\xi) d\xi. \quad (15)$$

from (14), (15) and (11), it follows that

$$u(t, x) + u_x(t, x) = e^x \int_x^{+\infty} e^{-\xi} y(\xi) d\xi \geq 0, \quad \text{if } x \geq q(t, x_0) \quad (16)$$

and

$$u(t, x) - u_x(t, x) = e^{-x} \int_{-\infty}^x e^{\xi} y(\xi) d\xi \leq 0, \quad \text{if } x \leq q(t, x_0). \quad (17)$$

From (16) and (17), we get $-u_x \leq |u|$ on $\mathbf{R}$.

Multiplying Eq.(1) by u and integrating by parts, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}} (u^2 + u_x^2) dx &= -\lambda \int_{\mathbf{R}} (u^2 + u_x^2) dx - a \int_{\mathbf{R}} u u_x u_{xx} dx - b \int_{\mathbf{R}} u^2 u_{xxx} dx \\ &= -\lambda \int_{\mathbf{R}} (u^2 + u_x^2) dx - \frac{2b-a}{2} \int_{\mathbf{R}} u_x^3 dx \\ &\leq -\lambda \int_{\mathbf{R}} (u^2 + u_x^2) dx + \frac{|a-2b|}{2} \|u\|_{L^1} \|u\|_{H^1}^2 \\ &\leq -\lambda \int_{\mathbf{R}} (u^2 + u_x^2) dx + \frac{|a-2b|}{2} e^{-\lambda t} \|u_0\|_{L^1} \|u\|_{H^1}^2. \end{aligned}$$

Applying the Gronwall's inequality, one has

$$\|u\|_{H^1}^2 \leq \|u_0\|_{H^1}^2 \exp\left(-2\lambda t + \frac{|a-2b|}{\lambda} \|u_0\|_{L^1} (1 - e^{-\lambda t})\right).$$

This completes the proof of Lemma 3.2.

Theorem 3.3. *Let $u_0 \in H^s(\mathbf{R}) \cap L^1(\mathbf{R})$, $s > \frac{3}{2}$ and $y_0 = u_0 - u_{0xx}$ satisfies*

$$y_0 \leq 0 \quad \text{on} \quad (-\infty, x_0] \quad \text{and} \quad y_0 \geq 0 \quad [x_0, \infty)$$

for some point $x_0 \in \mathbf{R}$. Then problem (6) has a global strong solution satisfying

$$u = u(\cdot, u_0) \in C([0, \infty); H^s(\mathbf{R})) \cap C^1([0, \infty); H^{s-1}(\mathbf{R})).$$

Remark. If x_0 goes to $\pm\infty$, when $\lambda = 0$ in Eq.(1), we recover the Theorem 3 obtained by Lai and Wu [8], where the assumption $y_0 \geq 0$ on $\mathbf{R}$ or $y_0 \leq 0$ on $\mathbf{R}$ is needed. Therefore, Theorem 3 in [8] is a special case of Theorem 3.3.

Before proving the theorem, we firstly consider the differential equation

$$\begin{cases} q_t = bu(t, q), & t \in [0, T), \quad x \in \mathbf{R}, \\ q(0, x) = x, & x \in \mathbf{R}, \end{cases} \quad (18)$$

where u solves Eq.(1) and $T > 0$.

Proof. Here, we assume $s = 3$ to prove the theorem. Let T be the maximal time of existence of the solution u to problem (6) with initial data $u_0 \in H^3$.

From (11) and the formulas (14)-(15), we deduce that

$$\begin{aligned} u_x(t, x) &= -u(t, x) + e^x \int_x^\infty e^{-\xi} y(t, \xi) d\xi \\ &\geq -u(t, x), \quad x \geq q(t, x_0) \end{aligned} \quad (19)$$

and

$$\begin{aligned} u_x(t, x) &= u(t, x) - e^{-x} \int_{-\infty}^x e^{\xi} y(t, \xi) d\xi \\ &\geq u(t, x), \quad x \leq q(t, x_0). \end{aligned} \quad (20)$$

Therefore, we have

$$u_x(t, x) \geq -\|u(t, x)\|_{L^\infty}, \quad (t, x) \in [0, T) \times \mathbf{R}. \quad (21)$$

From the Sobolev imbedding theorem and Lemma 3.2, we get

$$\begin{aligned} \|u(t, x)\|_{L^\infty}^2 &\leq \|u(t, x)\|_{H^1}^2 \\ &\leq \|u_0\|_{H^1}^2 \exp\left(-2\lambda t + \frac{|a-2b|}{\lambda} \|u_0\|_{L^1} (1 - e^{-\lambda t})\right). \end{aligned} \quad (22)$$

It follows from (21)-(22) that

$$\begin{aligned} u_x(t, x) &\geq - \| u(t, x) \|_{L^\infty} \\ &\geq - \| u_0 \|_{H^1} \exp \left(- \lambda t + \frac{|a - 2b|}{2\lambda} \| u_0 \|_{L^1} (1 - e^{-\lambda t}) \right), \end{aligned} \quad (23)$$

where $(t, x) \in [0, T) \times \mathbf{R}$. The above inequality (23) and Theorem 2.2 imply that $T = \infty$. This proves that the solution $u(t, x)$ exists globally in time.

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