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Pricing Variance Swaps with Stochastic Volatility

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Abstract

In this paper we present an efficient approach to price variance swaps with continuous time. We also obtain a closed form exact solution based on the Heston stochastic volatility model. This is accomplished by deriving an explicit valuation method using risk-neutral probability. We solve for the price of the simple variance swap in an ordinary differential equation assuming no arbitrage. In addition, we develop a partial differential equation under the assumption that the price function of the underlying assets have second order smooth partial derivatives.

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Key-words: Variance Swaps, Stochastic Volatility, Heston Model

1 Introduction

A variance swap is a derivative contract which allows investors to trade future realized volatility against current implied volatility. These derivative contracts

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are fundamental to the world of finance given that variance of returns are the standard measures of encapsulating risk. The interest from investors, banks, exchanges and regulators with respect to pricing variance swaps has gathered extensive momentum due to events in the last 15 years, including the financial, euro crises and brexit. According to the Black and Scholes model [1], the volatility of a stock is revealed by the market price of an option on the stock. However, the model makes the assumption that volatility is constant, which contradicts the empirical observation that, for a given expiry, the market prices of option contracts at different strikes typically imply different Black-Scholes volatility parameters (see among others, Britten -Jones and Neuberger [3]). Regarding the risk-neutral distribution of the path-dependent realized volatility, researchers are able to extract information from the profile of European option prices at a given expiry.

Prior research has priced variance swaps with stochastic volatility in discrete time. Zhu and Lian [8] find a closed form solution based on the Heston [6] two factor stochastic volatility model. Zhu and Lian [9] introduce stochastic volatility through simultaneous jumps in both the asset price and volatility processes, and Cui et al [5] allow changes in volatility via regime switching. Clearly, pricing variance swaps with stochastic volatility in discrete time has improved the accuracy since the Black and Scholes constant volatility model. However, given that volatility is constantly changing in financial markets on an intraday basis, the pricing of variance swaps can be enhanced if we are able to have a pricing model with stochastic volatility in continuous time. Such models are also found in Brockhaus and Long [2], Carr et al. [4], and Swishchuk [7]).

In this paper, we present an approach to price variance swaps with continuous time. We obtain a closed-form exact solution for the partial differential equation system based on the Heston stochastic volatility model [6]. Under some assumptions including an independence condition, the distribution of realized variance determines the value of a stock option.

The remainder of the paper is organized as follows: The next section introduces the notations used in the pricing. Section 3 prices the variance swaps with stochastic volatility in continuous time. Finally, section 4 concludes by discussing possible shortcomings and avenues for future research.

2 Assumptions and Notation

Consider a probability space $(\Omega, \mathcal{F}, P)$ and a time interval $t = [0, T]$, where $\mathcal{F}_t$ is a filtration. Let the underlying share price S be a stochastic process adapted to the filtration $\mathcal{F}_t$. This means that S_t can be determined by the information available at time t. The return process X_t is a function of S_t and the variation in X is denoted by $\langle X \rangle$. This is known as the realized variance of the returns on S. We denote by B_t a Brownian motion, which is the limit, as $\triangle_t \to 0$, of a Gaussian discrete-time random walk

$$B_{t+\triangle_t} = B_t + \epsilon_t \sqrt{\triangle_t},$$

where ϵ_t are i.i.d standard normal distribution, $\epsilon_t \sim N(0,1)$. Let $T > 0$ be an arbitrary time horizon. We introduce the following assumptions.
(1) Constant real interest rate r.
(2) Arbitrage-free markets.
(3) The underlying share price S satisfies

$$\frac{dS_t}{S_t} = rdt + \sigma_t dB_t, \quad S_t > 0,$$

where B_t is a Brownian motion with standard deviation σ_t.
(4) The variance of the share price σ_t^2 satisfies

$$d\sigma_t^2 = \kappa(\bar{\sigma}^2 - \sigma^2)dt + \gamma\rho\sigma_t dB_t + \gamma\sqrt{1-\rho^2}\sigma_t dB_t^\sigma.$$

(5) Let

$$\int_t^T \sigma_\tau^2 d\tau \quad \text{be finite.}$$

(6) Let B_t and B_t^σ be independent processes such that

$$dB_t dB_t^\sigma = 0.$$

We now derive the pricing formulae.

3 Efficient Pricing

By the no-arbitrage assumption (2), we know on a filtered probability space $(\Omega, \mathcal{F}, P)$, there exists an equivalent probability measure Q such that $\forall\, t \le T$, a derivative paying at time T, S_T has time-t price equal to $\mathbb{E}_t^Q S_T$, where $\mathbb{E}_t^Q$ denotes the $\mathcal{F}_t$-conditional Q-expectation. We emphasize that we will not derive formulae under the actual physical probability measure P. In our model, all expectations are with respect to the risk-neutral measure Q. Let h be the value of the variance swaps and g be the price of the variance swaps. Because of the absence of arbitrage, the price of the swap must make the payment of variance equal to the price of the contract with probability 1 under the physical probability measure P. Since P and Q are all probability measures, they must agree on all events of probability 1. Thus, the pricing problem must satisfy

$$\mathbb{E}^Q h(\langle X \rangle_T) = \mathbb{E}^Q g(S_T).$$

Let's consider a variance swap paying a dollar amount

$$h(\langle X \rangle_T) = \int_t^T \sigma_\tau^2 d\tau,$$
$$g(S_T) = K_t^2.$$

The payment at time T is hence

$$\int_t^T \sigma_\tau^2 d\tau - K_t^2.$$

We will derive the the strike price of the swap, where K_t is the price that makes the market value of the swap at time t is zero.

The strike price of the swap must be such that

$$K_t^2 = \mathbb{E}_t^Q \int_t^T \sigma_\tau^2 d\tau,$$

where E_t^Q denotes the expectation under the risk-neutral measure. To compute the strike price, we integrate the variance of the swap.

$$\sigma_\tau^2 = \sigma_t^2 - \int_t^T \kappa(\sigma_s^2 - \bar{\sigma}^2)ds + \gamma\rho \int_t^T \sigma_s dB_s + \gamma\sqrt{1-\rho^2} \int_t^T \sigma_s dB_s^\sigma.$$

The expectation satisfies

$$\mathbb{E}_t^Q[\sigma_\tau^2] = \sigma_t^2 - \mathbb{E}_t^Q \int_t^T \kappa(\sigma_s^2 - \bar{\sigma}^2)ds \qquad = \sigma_t^2 - \int_t^T \kappa(\mathbb{E}_t^Q(\sigma_\tau^2 - \bar{\sigma}^2))dt,$$

where we have used the fact that

$$\mathbb{E}_t^Q \int_t^T \sigma_s dB_s = 0,$$
$$\mathbb{E}_t^Q \int_t^T \sigma_s dB_s^\sigma = 0.$$

Denote $\mathbb{E}_t^Q[\sigma_\tau^2]$ as y_τ, $\tau \geq t$, then we obtain the ODE

$$y_\tau^{'} = -\kappa(y_\tau - \bar{\sigma}^2), \qquad (1)$$
$$y_t = \sigma_t^2.$$

Solving the ODE in equation (1) yields

$$E_t^Q[\sigma_\tau^2] = \bar{\sigma}^2 + e^{-\kappa(\tau-t)}(\sigma^2 - \bar{\sigma}^2)$$

from which we obtain the strike price

$$K_t^2 = E_t^Q \int_t^T \sigma_\tau^2 = \bar{\sigma}^2(T-t) + \frac{1}{\kappa}(\sigma_t^2 - \bar{\sigma}^2)(1 - e^{-\kappa(\tau - t)}). \tag{2}$$

In a more general situation, we have a security that pays $g(S_T)$ at time T, where g is a sufficiently smooth function of S_T. We now derive the pricing partial differential equation (PDE).

Let $f(t, S_t, \sigma_t^2)$ be the price of the security with well-defined and continuous first and second order partial derivatives. Under Q the expected return on any asset equals the risk-free rate. Hence,

$$\mathbb{E}_t^Q \left[\frac{df(t, S_t, \sigma_t^2)}{f(t, S_t, \sigma_t^2)} \right] = r dt$$

By Ito's lemma, the drift under Q is

$$\begin{aligned} E_t^Q \left[df\left(t, S_t, \sigma_t^2\right) \right] = &\left(f_t' + rS_t f_S' + \left[\kappa \left(\bar{\sigma}^2 - \sigma_t^2 \right) \right] f_{\sigma^2}' + \right. \\ &\left. + \frac{1}{2}\sigma_t^2 S_t^2 f_{SS}'' + \frac{1}{2}\gamma^2 \sigma_t^2 f_{\sigma^2\sigma^2}'' + \rho\gamma\sigma_t^2 S_t f_{S\sigma^2}'' \right) dt \end{aligned}$$

Thus, the PDE is

$$-rf\left(t, S_t, \sigma_t^2\right) + E_t^Q \left[df\left(t, S_t, \sigma_t^2\right) \right] = 0 \tag{3}$$

with boundary condition

$$f(T, S_T, \sigma_T^2) = g(S_T). \tag{4}$$

4 Conclusion

In this paper, we contribute to the variance swap literature by deriving an explicit valuation method using risk-neutral probability. Under the absence of arbitrage assumption, we solve for the price of the simple variance swap in an ordinary differential equation. Moreover, we derive a more general pricing partial differential equation under the assumption that the price function of the underlying assets have second order smooth partial derivatives.

The development of a variance swap pricing model with stochastic volatility will produce better empirical results, given that stock volatility varies continuously through time. Our findings, will stimulate interesting avenues for further research. These could include extending the dynamics and risks of the underlying asset.

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