On the Equicontinuity of Separately Equicontinuous Sets of Mappings

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Abstract

A criterion for the equicontinuity of certain separately equicontinuous sets of mappings, whose validity depends strongly on a version of the Banach-Steinhaus theorem in the context of linearly topologized modules, is obtained.

Mathematics Subject Classification: 13F30, 16W80, 46H25

Keywords: discrete valuation rings, linearly topologized modules, equicontinuity

1 Introduction and preliminaries

Bourbaki [1, p. 28] established a criterion for the equicontinuity of certain separately equicontinuous sets of mappings in whose proof a general version of the Banach-Steinhaus theorem in the framework of locally convex spaces plays a central role. In this note we apply the same argument in order to prove a criterion for the equicontinuity of certain separately equicontinuous sets of
mappings in whose proof a version of the Banach-Steinhaus theorem in the framework of linearly topologized modules is also quite relevant.

In this note ring will mean commutative ring with an identity element $1 \neq 0$ and all modules under consideration will be unitary modules. A principal ring $R$ is said to be a discrete valuation ring [5, Chap. I] if the set $M$ of non-invertible elements of $R$ constitutes a non-trivial ideal of $R$ (thus $R$ is not a field). If $\pi$ is a generator of $M$,

$$\{\pi^n R ; n = 1, 2, \ldots\}$$

is a fundamental system of neighborhoods of 0 in $R$ consisting of ideals of $R$ such that $\bigcap_{n \geq 1} \pi^n R = \{0\}$, and hence $R$ is a metrizable linearly topologized ring. In what follows $R$ will denote an arbitrary discrete valuation ring, unless otherwise specified. $E$ is said to be a linearly topologized $R$-module if it is a topological $R$-module whose origin admits a fundamental system of neighborhoods consisting of submodules of $E$ [6, §31; 3]. It is clear that a subset $B$ of a linearly topologized $R$-module $E$ is bounded if and only if for each neighborhood $U$ of 0 in $E$ which is a submodule of $E$ there exists an integer $k \geq 1$ so that $\pi^k B \subset U$.

## 2 The result

We shall prove the following

**Theorem 2.1** Let $T$ be a metrizable topological space, $E$ a barrelled metrizable linearly topologized $R$-module and $G$ an arbitrary linearly topologized $R$-module. Let $\mathcal{M}$ be a set of mappings from $E \times T$ into $G$ satisfying the following assumptions:

(a) for each $t \in T$,

$$\{u \in E \mapsto f(u, t) \in G ; f \in \mathcal{M}\}$$

is an equicontinuous set of $R$-linear mappings from $E$ into $G$;

(b) for each $x \in E$,

$$\{v \in T \mapsto f(x, v) \in G ; f \in \mathcal{M}\}$$

is an equicontinuous set of mappings from $T$ into $G$.

Under these conditions, $\mathcal{M}$ is equicontinuous.

**Proof:** By (4), p. 172 of [2], it suffices to prove that, for each $(x, t) \in E \times T$ and for each sequence $((x_n, t_n))_{n \geq 1}$ in $E \times T$ converging to $(x, t)$ in the metrizable topological space $E \times T$, $(f(x_n, t_n))_{n \geq 1}$ converges uniformly to $f(x, t)$ for
$f \in \mathcal{M}$. Indeed, let $W$ be a neighborhood of 0 in $G$ which is a submodule of $G$ and let us show that the set

$$\{u \in E \mapsto f(u, t_n) - f(u, t) \in G \ ; \ n \geq 1, \ f \in \mathcal{M}\}$$

of continuous $R$-linear mappings from $E$ into $G$ is equicontinuous. But, since $E$ is barrelled, it is enough to show that the just-mentioned set is simply bounded in view of Theorem 3.4 of [4]. For this purpose, let $u \in E$ be fixed. By (b) and the fact that $(t_n)_{n \geq 1}$ converges to $t$ in $T$, there exists an integer $m > 1$ so that $f(u, t_n) - f(u, t) \in W$ for all $n > m$ and for all $f \in \mathcal{M}$. On the other hand, the equicontinuity of the set

$$\{z \in E \mapsto f(z, t_n) - f(z, t) \in G \ ; \ n = 1, \ldots, m, \ f \in \mathcal{M}\}$$

of $R$-linear mappings from $E$ into $G$, furnished by (a), implies its pointwise boundedness. Consequently, there is an integer $l \geq 1$ so that

$$\pi^l \{f(u, t_n) - f(u, t) ; \ n = 1, \ldots, m, \ f \in \mathcal{M}\} \subset W.$$

But, as $W$ is a submodule of $G$, what we have seen yields

$$\pi^l \{f(u, t_n) - f(u, t) ; \ n > m, \ f \in \mathcal{M}\} \subset \pi^l W \subset W.$$

Therefore

$$\pi^l \{f(u, t_n) - f(u, t) ; \ n \geq 1, \ f \in \mathcal{M}\} \subset W.$$

By the arbitrariness of $u$ and what we have observed, the equicontinuity of the set

$$\{u \in E \mapsto f(u, t_n) - f(u, t) \in G \ ; \ n \geq 1, \ f \in \mathcal{M}\}$$

is guaranteed. Moreover, by (a), the set

$$\{u \in E \mapsto f(u, t) \in G ; \ f \in \mathcal{M}\}$$

of $R$-linear mappings from $E$ into $G$ is equicontinuous. Hence the set

$$\{u \in E \mapsto f(u, t_n) \in G ; \ n \geq 1, \ f \in \mathcal{M}\}$$

of $R$-linear mappings from $E$ into $G$ is equicontinuous and, since the sequence $(x_n - x)_{n \geq 1}$ converges to 0 in $E$, there is an integer $r \geq 1$ so that $f(x_n, t_n) - f(x, t_n) \in W$ for all $n \geq r$ and for all $f \in \mathcal{M}$. Moreover, by (b), there is an integer $s \geq 1$ so that $f(x, t_n) - f(x, t) \in W$ for all $n \geq s$ and for all $f \in \mathcal{M}$. Thus

$$f(x_n, t_n) - f(x, t) = [f(x_n, t_n) - f(x, t_n)] + [f(x, t_n) - f(x, t)] \in W + W \subset W$$

for all $n \geq \max \{r, s\}$ and for all $f \in \mathcal{M}$, thereby concluding the proof.
An immediate consequence of Theorem 2.1, which is precisely Proposition 3.8 of [4], reads:

**Corollary 2.2** Let $E, F$ be metrizable linearly topologized $R$-modules, with $E$ barrelled, and let $G$ be an arbitrary linearly topologized $R$-module. Then every separately equicontinuous set of $R$-bilinear mappings from $E \times F$ into $G$ is equicontinuous.

The example below shows that the barrelledness of $E$ is essential for the validity of Corollary 2.2.

**Example 2.3** Let us consider the submodule $E = R^{(\mathbb{N})}$ of the $R$-module $R^{\mathbb{N}}$, endowed with the topology induced by the product topology on $R^{\mathbb{N}}$, under which $E$ is a metrizable linearly topologized $R$-module. Since $\pi E$ is a barrel in $E$ which is not a neighborhood of 0 in $E$, $E$ is not barrelled. Moreover, the separately continuous $R$-bilinear mapping

$$A : ((\lambda_n)_{n \geq 1}, (\mu_n)_{n \geq 1}) \in E \times E \mapsto \sum_{n \geq 1} \lambda_n \mu_n \in R$$

is discontinuous. In fact, if $k, m_1, \ldots, m_k$ are arbitrary integers $\geq 1$,

$$\alpha = (0, \ldots, 0, 1, 0, \ldots, 0, \ldots) \in ((\pi^{m_1} R) \times \ldots \times (\pi^{m_k} R) \times R \times \ldots \times R \times \ldots) \cap E$$

and $A(\alpha, \alpha) = 1 \notin \pi R$.

Finally, an immediate consequence of Proposition 2.5 of [4] and Theorem 2.1 reads:

**Corollary 2.4** Let $E$ be a complete metrizable linearly topologized $R$-module such that $\pi^n U$ is a neighborhood of 0 in $E$ for every integer $n \geq 1$ and for every neighborhood $U$ of 0 in $E$. If $T, G$ and $\mathcal{M}$ are as in the statement of Theorem 2.1, then $\mathcal{M}$ is equicontinuous.

One may also mention that the condition “$\pi^n U$ is a neighborhood of 0 in $E$ for every integer $n \geq 1$ and for every neighborhood $U$ of 0 in $E$” is essential for the validity of Corollary 2.4, as the example below shows.

**Example 2.5** Assume that $R$ is complete, let $E$ be the $R$-module $R^\mathbb{N}$ endowed with the product topology, under which $E$ is a complete metrizable linearly topologized $R$-module (note that $U = \pi R \times R \times R \times \ldots \times R \times \ldots$ is a neighborhood of 0 in $E$, but $\pi U$ is not), and let $F$ be the submodule $R^{(\mathbb{N})}$ of $R^\mathbb{N}$ endowed with the linear $R$-module topology induced by that of $E$. 
Then, by arguing as in Example 2.3, one sees that the separately continuous $R$-bilinear mapping
\[
((\lambda_n)_{n \geq 1}, (\mu_n)_{n \geq 1}) \in E \times F \mapsto \sum_{n \geq 1} \lambda_n \mu_n \in R
\]
is discontinuous.

\section*{References}


\textbf{Received: April 29, 2022; Published: May 20, 2022}