The Relation Between the First and Second Multipliers of a Semivalue

Dinamérico P. Pombo Jr.

Instituto de Matemática e Estatística
Universidade Federal Fluminense
Rua Professor Marcos Waldemar de Freitas Reis, s/n²
Bloco G, Campus do Gragoatá
24210-201 Niterói, RJ Brasil

Abstract

In this note the first multiplier $m$ of a semivalue $\cdot$ on a commutative ring with an identity element is introduced, and the relation $m = \max\left\{1, \frac{M}{2}\right\}$ between $m$ and the second multiplier $M$ of $\cdot$ is established.

Mathematics Subject Classification: 13A18, 13F30, 12J20

Keywords: commutative ring, semivalue, first multiplier, second multiplier

1. Introduction

Let $R$ be a commutative ring with an identity element $1 \neq 0$. A mapping $\cdot : R \to \mathbb{R}_+$ is a semivalue on $R$ [2] if the following conditions are satisfied:

(a) $|0| = 0$ and $|1| = 1$;

(b) $|\lambda \mu| = |\lambda| |\mu|$ for all $\lambda, \mu \in R$;

(c) there exists a $C \in \mathbb{R}$ so that $|\lambda + \mu| \leq C \max\{||\lambda|, |\mu|\}$ for all $\lambda, \mu \in R$.

By (a) and (c), $C \geq 1$. 

This article is distributed under the Creative Commons by-nc-nd Attribution License. Copyright © 2020 Hikari Ltd.
If \( P \) is a non-trivial prime ideal of a commutative ring \( R \) with an identity element \( 1 \neq 0 \), \( R \), the mapping \(| \cdot |_P\) defined by \(|\lambda|_P = 0\) if \( \lambda \in P \) and \(|\lambda|_P = 1\) if \( \lambda \in R \setminus P \) is an absolute semivalue on \( R \) that is not an absolute value [7, p. 139]. In particular, for each prime integer \( q \), \(| \cdot |_q\) is an absolute semivalue on \( \mathbb{Z} \) that is not an absolute value. On each discrete valuation ring there is a natural absolute value (hence semivalue) [6, Chapter I]. Each valuation on a field \( K \) [1, p. 3; 4, p. 403] is a semivalue on \( K \).

The second multiplier \( M \) of an arbitrary semivalue \(| \cdot |\) on \( R \) is the real number

\[
M = \inf \{ C; C \in \mathbb{R} \text{ and } |\lambda + \mu| \leq C \max\{|\lambda|, |\mu|\} \text{ for all } \lambda, \mu \in R \}.
\]

It is obvious that \(|\lambda + \mu| \leq M \max\{|\lambda|, |\mu|\} \text{ for all } \lambda, \mu \in R \). We have proved in [2] that

\[
M = \max\{1, |2|\}.
\]

2. The result

Let \( R \) be an arbitrary commutative ring with an identity element \( 1 \neq 0 \) and \(|\cdot|: R \to \mathbb{R}_+\) a mapping satisfying conditions (a) and (b). Now let us introduce condition

\((c')\) there exists a \( d \in \mathbb{R} \) so that \(|\lambda + \mu| \leq d(|\lambda| + |\mu|)\) for all \( \lambda, \mu \in R \),

which has already been considered in [5, p. 51] when \( R \) is a field.

As above, \( d \geq 1 \). It is easily seen that the validity of (c) is equivalent to that of \((c')\).

By definition, the first multiplier \( m \) of an arbitrary semivalue \(| \cdot |\) on \( R \) is the real number

\[
m = \inf \{ d; d \in \mathbb{R} \text{ and } |\lambda + \mu| \leq d(|\lambda| + |\mu|) \text{ for all } \lambda, \mu \in R \}.
\]

We have that

\[
|\lambda + \mu| \leq m(|\lambda| + |\mu|) \text{ for all } \lambda, \mu \in R
\]

and

\[
1 \leq m \leq M \leq 2m \quad (\text{hence } m = 1 \text{ if } M = 1).
\]

The purpose of this note is to extend Proposition 3, p. 56 of [5] (proved for valuations on fields) to our context, by using the argument of the proof of that proposition. More precisely, we shall establish the following

**Theorem 2.1.** If \(| \cdot |\) is a semivalue on \( R \),

\[
m = \max \left\{ 1, \frac{M}{2} \right\}.
\]
The relation between the first and second multipliers of a semivalue

Proof. Let us put \( \ell = \max \left\{ 1, \frac{M}{2} \right\} \). Since \( M \leq 2\ell \), one has

\[
|\lambda + \mu| \leq 2\ell \max \{ |\lambda|, |\mu| \}
\]

for all \( \lambda, \mu \in R \). Consequently, by induction,

\[
|\lambda_1 + \lambda_2 + \cdots + \lambda_{2^r}| \leq 2^r \ell^r \max\{ |\lambda_1|, |\lambda_2|, \ldots, |\lambda_{2^r}| \}
\]

for every integer \( r \geq 1 \) and \( \lambda_1, \lambda_2, \ldots, \lambda_{2^r} \in R \). In particular, \( |2^r| \leq 2^r \ell^r \) for every integer \( r \geq 1 \).

Let \( r \) be an integer \( \geq 1 \), \( n = 2^r \) and \( \lambda, \mu \in R \setminus \{0\} \).

Claim 1. \( |\lambda + \mu| \leq 2n \ell^{n+r-1} (|\lambda| + |\mu|)^{n-1} \).

Indeed,

\[
|\lambda + \mu|^{n-1} = |(\lambda + \mu)^{n-1}|
\]

\[
= \left| \lambda^{n-1} + \binom{n-1}{1} \lambda^{n-2} \mu + \binom{n-1}{2} \lambda^{n-3} \mu^2 + \cdots + \binom{n-1}{n-3} \lambda^2 \mu^{n-3} + \binom{n-1}{n-2} \lambda \mu^{n-2} + \mu^{n-1} \right|
\]

\[
\leq 2^r \ell^r \max_{0 \leq k \leq n-1} |\binom{n-1}{k} \lambda^{(n-1)-k} \mu^k| = n\ell^r \max_{0 \leq k \leq n-1} \left| \binom{n-1}{k} \right| |\lambda|^{(n-1)-k} |\mu|^k.
\]

Before proceeding, let us prove

Claim 2. For \( r = 1, 2, \ldots \) and \( 0 \leq p < 2^r \), one has

\[
|p| \leq 2p \ell^r.
\]

We shall argue by induction on \( r \geq 1 \), the assertion being obvious for \( r = 1 \). Let \( r > 1 \) and assume the validity of the assertion for \( r-1 \). We need to show that \( |p| \leq 2p \ell^r \) if \( 0 \leq p < 2^r \), which holds if \( 0 \leq p < 2^{r-1} \). Let us assume that \( 2^{r-1} \leq p < 2^r \) and write \( p = 2^{r-1} + (p - 2^{r-1}) \). Then

\[
|p| \leq 2\ell \max\{ |2^{r-1}|, |p - 2^{r-1}| \} = \max\{2\ell|2^{r-1}|, 2\ell|p - 2^{r-1}| \}.
\]

If \( |p| \leq 2\ell|2^{r-1}| \), since \( |2^{r-1}| \leq 2^{r-1} \ell^{r-1} \), it follows that

\[
|p| \leq 2\ell 2^{r-1} \ell^{r-1} = 2^r \ell^r \leq 2p \ell^r,
\]

because \( 2^r \leq 2p \). And, if \( |p| \leq 2\ell|p - 2^{r-1}| \), since \( 0 \leq p - 2^{r-1} < 2^r - 2^{r-1} = 2^{r-1} \), the inductive hypothesis furnishes

\[
|p - 2^{r-1}| \leq 2(p - 2^{r-1}) \ell^{r-1}.
\]
Consequently, since \(2(p - 2^{r-1}) < p\), it follows that

\[|p| \leq 2\ell|p - 2^{r-1}| \leq (2\ell)2(p - 2^{r-1})\ell^{r-1} < 2\ell p\ell^{r-1} = 2p\ell^r.\]

Thus the proof of Claim 2 is finished.

Now let us conclude the proof of Claim 1. Indeed, the equality

\[2^{n-1} = \frac{n-1}{k=0} \binom{n-1}{k}\]

and Claim 2 imply that

\[|\lambda + \mu|^n \leq n\ell^r \max_{0 \leq k \leq n-1} \left| \binom{n-1}{k} \right| |\lambda|^{(n-1)-k} |\mu|^k \]

\[\leq 2n\ell^r \ell^{n-1} \left( \sum_{k=0}^{n-1} \binom{n-1}{k} |\lambda|^{(n-1)-k} |\mu|^k \right) = 2n \ell^{n+r-1} (|\lambda| + |\mu|)^{n-1},\]

proving Claim 1. Therefore

\[|\lambda + \mu| \leq 2^{r+1} \ell^{2^{r+1}-1} (|\lambda| + |\mu|)\]

for all integer \(r \geq 1\), and from the facts that

\[\lim_{r \to \infty} 2^{r+1} \ell^{2^{r+1}-1} = 1\] and \[\lim_{r \to \infty} \ell^{2^{r+1}-1} = \ell\]

one arrives at

\[|\lambda + \mu| \leq \ell (|\lambda| + |\mu|).\]

Consequently, \(m \leq \ell\). On the other hand, \(m \geq 1\) and \(m \geq \frac{M}{2}\), and hence \(\ell = \max \left\{ 1, \frac{M}{2} \right\} \leq m\). Thus \(m = \max \left\{ 1, \frac{M}{2} \right\}\), and the proof is complete.

**Remark 2.2.** We have already observed that \(m = 1\) if \(M = 1\). On the other hand, if \(M > 1\), the result mentioned in the introduction gives \(M = |2|\); consequently, by Theorem 2.1, \(m = 1\) if \(1 \leq |2| \leq 2\) and \(m = \frac{|2|}{2}\) if \(|2| > 2\).

Our last result has already been proved in [3] by means of another argument.

**Corollary 2.3.** In order that a semivalue \(|\cdot|\) on \(R\) be an absolute semivalue on \(R\) (that is, in order that \(|\lambda + \mu| \leq |\lambda| + |\mu|\) for all \(\lambda, \mu \in R\)), it is necessary and sufficient that \(C = 2\) be permissible for \(|\cdot|\).
The relation between the first and second multipliers of a semivalue

Proof. Since the necessity is clear, let us turn to the sufficiency. Indeed, if $C = 2$ is permissible for $|\cdot|$, $M \leq 2$. Hence, by Theorem 2.1, $m = \max \left\{ 1, \frac{M}{2} \right\} = 1$, and $|\cdot|$ is an absolute semivalue on $R$.

References


Received: April 5, 2020; Published: April 23, 2020