The Trapezoidal Fuzzy Number

Linear Programming

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Abstract

Linear Programming (LP) problem is one of optimization problems. Based on its limited resources and other restrictions, we find the optimal solution for the problem. LP problems have very wide applications in our daily problems. However, in practice, these LPs often fail to represent the real solutions. Such failures can be caused by some tight modeling assumptions. One attempt to address this failure is to replace the classical set into fuzzy sets. In this case, we call it Fuzzy Linear Programming.

There are some types of fuzzy LP problems. One type is the right sides of the constraints are fuzzy numbers. The other type is the coefficients of the objective function are the fuzzy numbers. The most complicated type is the right side, the coefficients of the variables and the coefficients of the objective function are fuzzy numbers. There are some types of fuzzy numbers. Two of them are trapezoidal fuzzy number and triangular fuzzy number. They are simple, easy to be counted and to be implemented. This research used trapezoidal fuzzy numbers.

There are some techniques to solve the fuzzy LP problems. In this research,
we will exploit ranking function introduced by Yager. Ranking function \( \mathcal{R} \) is a mapping from a family of fuzzy numbers, which is denoted by \( F(\mathbb{R}) \), into real number \( \mathbb{R} \). This paper propose a method to solve trapezoidal fuzzy number linear programming problem. To solve this problem, we use trapezoidal fuzzy numbers. By the properties of the operations which are defined on the linear ranking function, we construct an algorithm to solve the trapezoidal fuzzy number linear programming.

**Keywords**: trapezoidal fuzzy number, ranking function, fuzzy linear programming problem

## 1 Introduction

Linear programming is a one of the most important operation research technique and has a very wide application especially related to the optimization problem. Linear programming is a mathematical modeling technique that is designed to optimize the use of limited resources. Linear programming was first introduced by George Dantzig in 1947. Linear programming involves an objective function (single or multi) and constraint functions. The objective function related to the objectives to be achieved. This function will be maximized when it relates to the profit, or minimized when it relates to the production costs to be incurred. The objective function is a function of several variables, called the decision variables. The objective function contains of some variables that are called decision variables. In fact, all of the decision variables have to meet to all of the inequalities (equalities) that are called constraint functions. Each linear programming has 3 kinds of parameters, namely the coefficients of the objective function, technique coefficients and the right hand side coefficients of the constraint. The entire coefficients are classic numbers (Crisp Set). Some techniques to find the optimal solution of linear programming has been introduced, either using the graphical method, the simplex method (classical) and its revisions, as well as using the Tora and Lindo programs. Linear programming has been widely applied to many real problems, but often failed to answer the question precisely. It is caused by the resources cannot be measured with certainty. We will obtain the precise solution if the model constructs based on the fuzzy sets.

The fuzzy decision theory was first introduced by Zadeh, in 1970. In fact, decision is an uncertainty or has an ambiguous property. The fuzzy decision theory is developed widely. It is happened on fuzzy linear programming theory too, as part of the fuzzy decision theory. Tanaka et al has introduced the fuzzy mathematical programming in general.

The formulation of fuzzy linear programming was first introduced by Zimmermann. In the development of several researchers develop into other various types of fuzzy linear programming as well as various types of approaches to find the solution. Deldago et al. [3], makes a general model of fuzzy linear
programming within the limits of technical coefficients fuzzy and fuzzy right side. Fung and Hu, [5], introduced the linear programming with the technique coefficients based on fuzzy numbers. Maleki et al, [9], use a ranking function to solve the problem of fuzzy linear programming. Verdegay define the dual problem through parametric linear program and shows that the problem of primal - dual fuzzy linear program has the same solution.

The number of variables which are involved on the LP problems only several variables. So if the linear programming problem involved many variables, it will cause more difficulty to solve this problem. Based on this situation has been introduced above, this paper aims to solve the fuzzy linear programming using ranking function and the fuzzy numbers which are involved are trapezoidal fuzzy number.

2 Theoretical Review

The LP problems aren’t always simple problems. It is because the linear programming are involved many constraints and variables that may not be solved by graphical method. Therefore, an algorithm of mathematical procedures is required to find solutions to these complex issues. The most widely used procedure is the Simplex Method.

2.1. Fuzzy Sets

Based on Klir, at al [7], a fuzzy set $A$ in $X$ is defined as a set of an ordered pair $A = \{(x, \mu_A(x))| x \in X\}$. $\mu_A(x)$ is called membership function of the fuzzy set. The membership function maps from $X$ into the closed interval $[0,1]$. Support of a fuzzy set $A$ is a set of all $x \in X$ with $\mu_A(x) > 0$. Core of a fuzzy set $A$ is a set of all $x \in X$ with $\mu_A(x) = 1$. A fuzzy set $A$ is called normal if the Core is not an empty set.

An $\alpha$-cut (level subset -$\alpha$) of a fuzzy set $A$ is a crisp set that is defined as $A^\alpha = \{x \in X| \mu_A(x) \geq \alpha\}$. A strong $\alpha$-cut (strong level subset-$\alpha$) is defined as $A^\alpha_\delta = \{x \in X| \mu_A(x) > \alpha\}$. A fuzzy set $A$ of $X$ is convex if for any $x, y \in X$ and $\delta \in [0,1]$ then: $\mu_A(\delta x + (1-\delta)y) \geq \min\{\mu_A(x), \mu_A(y)\}$. One of the properties of a convex fuzzy set i.e. a fuzzy set is convex if and only if all of the non empty $\alpha$-cut is convex. A fuzzy number $A$ is a normal and convex fuzzy set. There are some kinds of fuzzy numbers. One of them is a trapezoidal fuzzy number. The definition of it is given as follow:

**Definition 2.1.** ([1],[4],[6],[7],[9],[10]) A fuzzy number $\tilde{A} = (a^L, a^U, \alpha_1, \alpha_2)$ is called a trapezoidal fuzzy number if its membership function meets the following mapping:

$$
  f(x) = \begin{cases} 
    \frac{x-(a^L-\alpha_1)}{\alpha_1}, & a^L-\alpha_1 \leq x \leq a^L \\
    1, & a^L \leq x \leq a^U \\
    \frac{(a^U+\alpha_2)-x}{\alpha_2}, & a^U \leq x \leq a^U+\alpha_2 \\
    0, & \text{others}
  \end{cases}
$$
The above definition can be illustrated as follow:

![Figure 1](image)

The support of the trapezoidal fuzzy number is \([a^l, a^u, \alpha_1, \alpha_2]\). The arithmetic operations on the set off all trapezoidal are following. For the trapezoidal fuzzy numbers \(\bar{A} = (a^l, a^u, \alpha_1, \alpha_2)\) and \(\bar{B} = (b^l, b^u, \beta_1, \beta_2)\) and \(r \in R\), then the arithmetic operations are defined as follow:

a. For \(r > 0\), then \(r\bar{A} = (ra^l, ra^u, r\alpha_1, r\alpha_2)\)
b. For \(r < 0\), then \(r\bar{A} = (ra^u, ra^l, -r\alpha_2, -r\alpha_1)\)
c. \(\bar{A} + \bar{B} = (a^l + b^l, a^u + b^u, \alpha_1 + \beta_1, \alpha_2 + \beta_2)\)
d. \(\bar{A} - \bar{B} = (a^l - b^l, a^u - b^u, \alpha_1 + \beta_1, \alpha_2 + \beta_2)\)

2.2. Ranking Function and Fuzzy Linear Programming

We will denote the set of all fuzzy numbers using \(F(R)\). Based on [1],[2],[4],[8],[11], a ranking function is a mapping \(\mathfrak{R}: F(R) \to \mathbb{R}\), from the set of fuzzy numbers into real number. For every two fuzzy numbers \(\bar{A}_1\) and \(\bar{A}_2\) in \(F(R)\), we are define relations as follow:

a. \(\bar{A}_1 \leq \bar{A}_2\) if and only if \(\mathfrak{R}(\bar{A}_1) \leq \mathfrak{R}(\bar{A}_2)\).
b. \(\bar{A}_1 > \bar{A}_2\) if and only if \(\mathfrak{R}(\bar{A}_1) > \mathfrak{R}(\bar{A}_2)\).
c. \(\bar{A}_1 = \bar{A}_2\) if and only if \(\mathfrak{R}(\bar{A}_1) = \mathfrak{R}(\bar{A}_2)\).

There are several ranking functions that have been investigated by several researchers. In this research, we use a linear ranking function: a ranking function \(\mathfrak{R}\) which meets \(\mathfrak{R}(k\bar{A}_1 + l\bar{A}_2) = k\mathfrak{R}(\bar{A}_1) + l\mathfrak{R}(\bar{A}_2)\) for every \(\bar{A}_1, \bar{A}_2 \in F(R)\). For a trapezoidal fuzzy number \(\bar{A} = (a^l, a^u, \alpha_1, \alpha_2)\), then according to Yager, the mapping by \(\mathfrak{R}\) is: \(\mathfrak{R}(\bar{A}) = \frac{1}{2}(a^l + a^u + \frac{1}{2}(\alpha_2 - \alpha_1))\).

3. Proposed Fuzzy Simplex Method

In this section will be discussed about a fuzzy linear programming problem with a trapezoidal fuzzy number. In this case, the fuzzy linear programming problem (fuzzy LP problem) has a general form as follow:

Maximized : \(\bar{Z} = c\bar{x}\)
such that : \(A\bar{x} = b, \bar{x} \geq 0\)  \hspace{1cm} (3.1)

where \(b \in (R)^m, x \in (R)^n, A \in R^{m \times n}\), and \(c^T \in (F(R))^n\).

We will develop the crisp simplex method into fuzzy simplex method. We propose the fuzzy simplex method which is based on the arithmetic operation of the trapezoidal fuzzy numbers, the properties of ranking functions and the classic
simplex method. The algorithm below is the algorithm to solve a standard minimum fuzzy LP problem.

1. Transform the fuzzy PL problem into a canonical form (the constraint must be positive, if necessary, change it into \( \leq \) relation by adding a slack variable).
2. Create an initial fuzzy simplex tableau, same as the crisp simplex tableau, only on its coefficient row, the objective function is a trapezoidal fuzzy number.
3. To test the optimality, we can see the value on the \( z_j - c_j \) row. If there is still a \( z_j - c_j < \bar{0} \), then is not optimal yet. We use the same way with \( \Re(z_j - c_j) < 0 \) to determine if \( z_j - c_j < \bar{0} \).
4. To fix the solution, choose an entering variable with the most negative \( \Re(z_j - c_j) \), and choose a leaving variable by choosing the most positive smallest \( R_i \).
5. On the new tableau, the element pivot must be transformed into 1, and the other elements on the same row must be transformed into 0 using elementary row operations. Repeat step 3 till the optimal solution is obtained.

By utilizing the fuzzy ranking numbers and the operations on trapezoidal fuzzy number, then we derive a simplex fuzzy method which is analogous to the classic simplex method. Given a fuzzy LP problem as follows:

Maximize \( \ddot{z} = (4,6,2,3)x_1 + (5,8,2,4)x_2 \)

such that

\[
\begin{align*}
3x_1 + 6x_2 & \leq 18 \\
5x_1 + 4x_2 & \leq 20 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Based on the standard maximum problem, the canonical form is as follow:

Maximize \( \ddot{z} = (4,6,2,3)x_1 + (5,8,2,4)x_2 + 0x_3 + 0x_4 \)

such that

\[
\begin{align*}
3x_1 + 6x_2 + x_3 & = 18 \\
5x_1 + 4x_2 + x_4 & = 20 \\
x_1, x_2, x_3, x_4 & \geq 0
\end{align*}
\]

<table>
<thead>
<tr>
<th>Iteration 0</th>
<th>( z_j )</th>
<th>(4,6,2,3)</th>
<th>(5,8,2,4)</th>
<th>( \ddot{0} )</th>
<th>( \ddot{0} )</th>
<th>( \ddot{R}_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 \backslash x_j )</td>
<td>( x_1 )</td>
<td>( x_2 )</td>
<td>( x_3 )</td>
<td>( x_4 )</td>
<td>( b_1 )</td>
<td>( R_i )</td>
</tr>
<tr>
<td>0</td>
<td>( x_1 )</td>
<td>3</td>
<td>6</td>
<td>1</td>
<td>0</td>
<td>18</td>
</tr>
<tr>
<td>0</td>
<td>( x_2 )</td>
<td>5</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>0</td>
<td>( x_3 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( z_j - c_j )</td>
<td>(4,6,2,3)</td>
<td>(-6,4,3,2)</td>
<td>(-8,5,4,2)</td>
<td>( \ddot{0} )</td>
<td>( \ddot{0} )</td>
<td></td>
</tr>
<tr>
<td>( \Re(z_j - c_j) )</td>
<td>( -5^+ )</td>
<td>( -7^+ )</td>
<td></td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

From the initial table above, the problem has a solution \( \ddot{z} = \ddot{0} \), for \( x_1 = 0, x_2 = 0, x_3 = 18, x_4 = 20 \), that is \( \ddot{z} = (4,6,2,3).0 + (5,8,2,4)0 + \ddot{0}.18 + \ddot{0}.20 = \ddot{0} \).

The table above shows that the solution has not optimum yet, because there is still \( \Re(z_j - c_j) < 0 \). It means \( z_j - c_j < \ddot{0} \). To fix the solution, firstly we have to
choose the entering variable by choosing the smallest negative value of $\Re(\tilde{z}_j - \tilde{c}_j)$, in this case $x_2$. Then we determine the value of $R_i = \frac{b_i}{a_{ij}^2}$ and choose the smallest positive value, in this case we choose $R_1 = 3$. The new basic variable $x_2$ replace the old basic variable, $x_3$. Element $a_{21}$ is called a pivot element. This element is located at the intersection of the row where leaving variable is located and the column of where the entering variable exists. The element pivot then turns into ‘1’ by using elementary row operation. The revised table becomes as follow:

<table>
<thead>
<tr>
<th>Iteration 1</th>
<th>$\tilde{c}_j$</th>
<th>(4,6,2,3)</th>
<th>(5,8,2,4)</th>
<th>0</th>
<th>0</th>
<th>$b_i$</th>
<th>$R_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5,8,2,4)</td>
<td>$x_2$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>$\frac{1}{7}$</td>
<td>0</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>$x_4$</td>
<td>3</td>
<td>0</td>
<td>$\frac{1}{7}$</td>
<td>1</td>
<td>8</td>
<td>$\frac{1}{5}$</td>
</tr>
<tr>
<td></td>
<td>$\tilde{z}_j$</td>
<td>(2, 4, 1, 2)</td>
<td>(5,8,2,4)</td>
<td></td>
<td>0</td>
<td>15, 24, 12, .6, 12.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\tilde{z}_j - \tilde{c}_j$</td>
<td>(-0,4,4)</td>
<td>(-3,3,6,6)</td>
<td></td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\Re(\tilde{z}_j - \tilde{c}_j)$</td>
<td>$\frac{1}{4}$</td>
<td>0</td>
<td>$\frac{1}{6}$</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Analogously, we obtain the new entering variable, $x_1$ and the leaving variable $x_4$. Furthermore, we get the next revised table as follow:

<table>
<thead>
<tr>
<th>Iteration 2</th>
<th>$\tilde{c}_j$</th>
<th>(4,6,2,3)</th>
<th>(5,8,2,4)</th>
<th>0</th>
<th>0</th>
<th>$b_i$</th>
<th>$R_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5,8,2,4)</td>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{7}$</td>
<td>$\frac{1}{3}$</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>(4,6,2,3)</td>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>$\frac{1}{7}$</td>
<td>$\frac{1}{3}$</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\tilde{z}_j$</td>
<td>(4,6,2,3)</td>
<td>(5,8,2,4)</td>
<td></td>
<td>(0, 12, 11, 12)</td>
<td>(3, 3, 6, 6)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\tilde{z}_j - \tilde{c}_j$</td>
<td>(-2,2,5,5)</td>
<td>(-3,3,6,6)</td>
<td></td>
<td>(0, 12, 11, 12)</td>
<td>(3, 3, 6, 6)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\Re(\tilde{z}_j - \tilde{c}_j)$</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{12}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From the above table, it shows that $\Re(\tilde{z}_j - \tilde{c}_j) \geq 0$, thus we obtain an optimal solution. The optimal solution is: $\tilde{z} = (\frac{57}{3}, \frac{88}{3}, \frac{26}{3}, \frac{44}{3})$ for $x_1 = \frac{8}{3}$ and $x_2 = \frac{5}{3}$.

What if the fuzzy LP has a ‘$\geq$’ on one of the constraints? In this case, besides from adding a slack variable, an artificial variable also must be added on the constraint, with the coefficient on the objective function is chosen such that the ($\tilde{z}_k - \tilde{c}_k$) becomes the most negative ones, with $\tilde{c}_k$ is the coefficient for the artificial variable.

Maximize \[ \tilde{z} = (1,5,2,4)x_1 + (10,12,2,6)x_2 \]
such that \[ 3x_1 + 10x_2 \leq 10 \]
\[ x_1 - x_2 \geq 2 \]
\[ x_1, x_2 \geq 0 \]
The canonical form is as follow:

Maximize \[ \tilde{z} = (1,5,2,4)x_1 + (10,12,2,6)x_2 + \bar{d}x_3 + \bar{d}x_4 - \bar{M}x_5 \]
such that \[ 3x_1 + 10x_2 + x_3 \leq 10 \]
\[ x_1 - x_2 + x_4 + x_5 \geq 2 \]
\[ x_1, x_2, x_3, x_4, x_5 \geq 0 \]

In this case, $\bar{M} = (m_1, m_2, a, b)$, with $b \geq 0$ and $m_1, m_2$ are very large numbers. Let $\bar{M} = (20, 26, 4, 8)$, then $\bar{M} = (-26, -20, 8, 4)$ and we get the initial simplex tableau as follow:
The trapezoidal fuzzy number linear programming

<table>
<thead>
<tr>
<th>Iteration 0</th>
<th>$c_i$</th>
<th>$(1,5,2,4)$</th>
<th>$(10,12,2,6)$</th>
<th>$b_i$</th>
<th>$(\tilde{-26}, -20, 8, 4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_i$</td>
<td>$x_1 \cdot y_1$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
<td>$x_4$</td>
</tr>
<tr>
<td>$\tilde{-26}$</td>
<td>0</td>
<td>3</td>
<td>10</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_1$</td>
<td>$(-26, -20, 8, 4)$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\tilde{z}_j - c_i$</td>
<td>$(-31, -21, 12, 6)$</td>
<td>0</td>
<td>20</td>
<td>26</td>
<td>4</td>
</tr>
<tr>
<td>Iteration 1</td>
<td>$c_j$</td>
<td>$(1,5,2,4)$</td>
<td>$(10,12,2,6)$</td>
<td>0</td>
<td>$(\tilde{-26}, -20, 8, 4)$</td>
</tr>
<tr>
<td>$\tilde{z}_j - c_j$</td>
<td>0</td>
<td>13</td>
<td>1</td>
<td>3</td>
<td>-3</td>
</tr>
<tr>
<td>$\tilde{z}_j - c_j$</td>
<td>$(-4, 4, 6, 6)$</td>
<td>$17, 11, 6$</td>
<td>0</td>
<td>5, 1, 4, 2</td>
<td>$(1,5,2,4)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Iteration 2</th>
<th>$c_j$</th>
<th>$(1,5,2,4)$</th>
<th>$(10,12,2,6)$</th>
<th>$b_j$</th>
<th>$R_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_j$</td>
<td>$x_1 \cdot y_1$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
<td>$x_4$</td>
</tr>
<tr>
<td>$(10,12,2,6)$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3/13</td>
<td>3/13</td>
</tr>
<tr>
<td>$(1,5,2,4)$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$-10/13$</td>
</tr>
<tr>
<td>$\tilde{z}_j - c_j$</td>
<td>$(-4, 4, 6, 6)$</td>
<td>$26, 46, 34$</td>
<td>$20, 26, 26$</td>
<td>$20, 26, 26$</td>
<td>$26, 26, 26$</td>
</tr>
</tbody>
</table>

By choosing a very large number for the coefficient of the artificial variable, this variable can be quickly out of the basis. On the last table above, the optimal solution is obtained.

Conclusions

Based on the results above, it can be concluded that to solve the fuzzy LP problems, with the coefficient of its objective function are trapezoidal fuzzy numbers, we can use the simplex method. The simplex method is based on the properties of fuzzy ranking functions and it is based on the trapezoidal fuzzy number operations and operations on fuzzy ranking function. In this research, the authors use linear fuzzy ranking and follow the solution introduced by Yager.

References


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