Quasicomponents of Elements and the Partition of a Finite Disconnected Topological Space

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Abstract

It is proved that the number of elements in the quasicomponents in a certain finite disconnected topological space divides the number of elements in the space.

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1 Introduction

The quasicomponent of a point \( x \) in a metric space \( X \) is defined as the intersection of all clopen (open and closed) subsets of \( X \) which contain \( x([2]) \). The objective of this paper is to see how this concept is broadened to general topological spaces with a view of getting a result regarding the number of elements in a certain finite topological space.

It is known that a topological space is connected if and only if the only subsets which are clopen are the empty set and the whole space (page 148 of [1]).

In order to ensure a supply of clopen sets therefore, an equivalence relation is defined on a disconnected space. It turns out that the equivalence classes emanating from this relation, which gives a partition of the space, coincides with the quasicomponent \( Q(x) \) for each \( x \in X \).
We conclude that the number of elements in the quasicomponent for a certain finite disconnected topological space divides the number of elements in it.

2 The Quasicomponent as an Equivalence Class

**Definition 2.1** Let $X$ be a topological space. The quasicomponent of $x \in X$ is defined by

$$Q(x) = \bigcap_{C \subset X \text{ clopen}} C$$

$x \in C$

**Lemma 2.2** Let $X$ be a disconnected topological space. Define a relation $\sim$ on $X$ by $x \sim y$ if and only if whenever $C$ is a clopen set containing $x$, it also contains $y$. Then $\sim$ is an equivalence relation.

Proof: Let $x \in X$ and let $x \in C$, $C$ a clopen set

$\Rightarrow x \sim x \forall x \in C$

$\Rightarrow \sim$ is reflexive

Suppose $x \sim y$

$\Rightarrow$ Any clopen set $C$ containing $x$ contains $y$.

Let $C'$ be a clopen set containing $y$.

We show that $x \in C'$

but suppose that $x \notin C'$

$\Rightarrow x \in C' = X - C'$

Now $C'$ is closed ($C$ is open)

and $C'$ is open ($C$ is closed)

$\Rightarrow C'$ is also clopen which contain $x$

$\Rightarrow y \in C'$ (since $x \sim y$)

This is a contradiction since $y \in C$

The contradiction arose from the assumption that $x \notin C$

Thus any clopen set which contains $y$ will also contain $x$

Therefore $x \sim y \Rightarrow y \sim x \forall x, y \in C$

$\Rightarrow \sim$ is symmetric

Let $x \sim y$ and $y \sim z$

Let $C$ be a clopen set $\ni x \in C$

$\Rightarrow y \in C$ ($x \sim y$)

$\Rightarrow z \in C$ ($y \sim z$)

Therefore $x \sim y, y \sim z \Rightarrow x \sim z$

$\Rightarrow \sim$ is transitive.

Hence $\sim$ is an equivalence relation.
Remark 2.3  

(i) \([x] = \{ t \in X | x \sim t \} \) is the equivalence class of \(x \in X\) under the relation \(\sim\).

(ii) The set of all equivalence classes under the relation \(\sim\) above is denoted by \(X/\sim\) and the number of equivalence classes by \(o(X/\sim)\).

(iii) The number of elements in an equivalence class for \(x \in X\) will be denoted by \(o([x])\). Likewise, \(o(X)\) and \(o(Q(x))\) will denote the number of elements in \(X\) and the quasicomponent of \(x \in X\) respectively.

Lemma 2.4  

\([x] = Q(x) \ \forall \ x \in X\)

Proof: Let \(y \in [x]\)

\(\Leftrightarrow x \sim y\)

\(\Leftrightarrow \) Any clopen set \(C \subset X\) which contains \(x\) must also contain \(y\)

\(\Leftrightarrow y \in \left( \bigcap_{\text{clopen } C \subset X \text{ with } x \in C} C \right)\)

\(\Leftrightarrow y \in Q(x)\)

Hence \([x] = Q(x)\).

Theorem 2.5  

If \(X\) is a finite disconnected topological space with \(o(X) = n\) and \(o(Q(x)) = o(Q(y)) = k \ \forall \ x, y \in X\). Then \(k\) divides \(o(X)\).

Proof: Since \(\sim\) is an equivalence relation on \(X\) (Lemma 2.2) which is finite with \(o(X) = n\), and \(Q(x) = [x] \ \forall \ x \in X\) (Lemma 2.4), it follows that the equivalence classes partitions \(X\). Suppose that \(o(X/\sim) = m\); we will have that \(n = km\).

References


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