On Unbounded Norm Demi KB-Operators
on Banach Lattices

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Abstract

By using unbounded norm topology, we investigate the notion of unbounded Kantorovich-Banach(KB)-operators and demicompactness of this type of operators.

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1 Introduction and preliminary notes

The many authors studied a kind of operators such as demicompact operators, Kantorovich-Banach(\(KB\))- operators, weakly compact and order weakly compact operators, weakly demicompact operators in Dodds\cite{7}, Meyer-Nieberg \cite{9}, Krichen et al\cite{8}, Petryshyn \cite{10}. 
In this article, we use the vector lattices theory to obtain a development to the unbounded norm demi \((KB)\)-operators, unbounded \(KB\)-operators. We investigate unbounded norm demi \(KB\)-operators and unbounded \((KB)\)-operators on normed vector lattices.

We use to the books [1], [9] for any knowledge on Riesz spaces and normed vector lattices.

Suppose that \(E\) is an ordered real linear vector space. We say that \(E\) is a vector lattice (Riesz space) if the infimum and supremum of \(\{x, y\}\) exist for every \(x, y \in E\). We use the notations called lattice operations \(\vee, \wedge\) for \(x \vee y = \sup\{x, y\}\) and \(x \wedge y = \inf\{x, y\}\). The positive part of a vector \(x \in E\) is defined by \(x^+ = x \vee 0\), the negative part of a vector \(x \in E\) is defined by \(x^- = -x \vee 0\) and the absolute value of a vector \(x \in E\) is defined by \(|x| = x^+ + x^-\). The set \(E^+ = \{x \in E : x \geq 0\}\) is called the cone of a Riesz space \(E\).

A seminorm \(p\) on a vector lattice \(E\) is called a lattice seminorm if \(p(x) \leq p(y)\) whenever \(|x| \leq |y|\) for all \(x, y \in E\). A seminorm \(p\) on a vector lattice \(E\) is called a lattice norm when \(p\) is a norm on \(E\). Then, we say that \((E, \|\cdot\|)\) is a normed vector lattice. We say that a Banach space is Banach lattice if its norm is a lattice norm. \(E'\) denotes the norm dual of \(E\). We say that Banach lattice \(E\) has an order continuous norm if \(x_n \downarrow 0 \in E\) implies \(\|x_n\| \downarrow 0\) for every sequence \((x_n)\) in \(E\). For example, the Banach lattice \(l^1\) of absolute summable sequences has order continuous norm, but the Banach lattice \(l^\infty\) of bounded sequences has no order continuous norm.

A Banach lattice \(E\) is called a (Kantorovich-Banach) \(KB\)-space if for all positive bounded increasing sequence \((x_n)\) in \(E\), \((x_n)\) converges. For example, \(AL\)-space, \(L_p\)-spaces for \(1 \leq p < \infty\), and so on. All \(KB\)-spaces have an order continuous norm. In general, the converse is not true. For example, the normed space \(c_0\) of all null sequences is not a \(KB\)-space, but, it has order continuous norm. The norm on a Banach lattice \(E\) is called \(p\)-additive if \(\|x + y\|^p = \|x\|^p + \|y\|^p\) holds for all disjoint \(x, y \in E\), where \(1 \leq p < \infty\). Any Banach lattice \(E\) having a \(p\)-additive norm is a \(KB\)-space. The dual \(E'\) of a Banach lattice \(E\) is a \(KB\)-space if and only if \(E'\) does not contain any sublattice isomorphic to \(c_0\). A mapping \(T : E \to X\) between a Banach lattice \(E\) and a Banach space \(X\) is said to be a \(KB\)-operator if for every positive norm bounded increasing sequence \((x_n)\) in \(E\), \((Tx_n)\) has a convergent subsequence in \(X\). Notice that the identity operator \(I : E \to E\) is a \(KB\)-operator on a Banach lattice \(E\) if and only if \(E\) is a \(KB\)-space. Let \(E\) be a Banach lattice and \(x, y \in E\). We define an order interval \([x, y]\) as the set \(\{z \in E : x \leq z \leq y\}\).

A subset in a vector lattice is called order bounded if it is contained in an order interval. A linear operator \(T\) from vector lattice \(E\) into another vector lattice \(F\) is called positive if \(0 \leq Tx\) whenever \(0 \leq x\) in \(E\). Here, an operator means bounded and linear operator.
2 Main Results

Definition 2.1 [3] Let $X$ be a Banach space. An operator $T : X \to X$ is called demicompact if, for every bounded sequence $(x_n)$ in $X$ such that $(x_n - Tx_n)$ converges in $X$, then there is a convergent subsequence of $(x_n)$.

Let us take a Banach space $X$ with infinite dimension. For the identity mapping $I$ on $X$, $-I$ is not compact, but it is demicompact, [3]. Clearly, every compact mapping is demicompact.

Definition 2.2 A net $\{x_\alpha\}$ in a vector lattice $L$ is called order convergent to an element $u \in L$ whenever there exists another net $\{y_\alpha\}$ of $L$ satisfying $|x_\alpha - u| \leq y_\alpha \downarrow 0$. The element $u$ is called the order limit of the net $\{x_\alpha\}$. The order limits are uniquely determined.

Definition 2.3 [7] An operator $T : E \to X$ between a Banach lattice $E$ and a Banach space $X$ is said to be order weakly compact if for each $x \in E^+$, the subset $T[0, x]$ is a relatively weakly compact subset of $X$.

Definition 2.4 [6] A sequence $(x_n)$ in $E$ is called unbounded order convergent to $x \in E^+$ if for every $u \in E^+$ the sequence $(|x_n - x| \wedge u)$ converges to zero in order. A sequence $(x_n)$ is called unbounded norm convergent to $x \in E^+$ if $(||x_n - x| \wedge u||)$ converges to zero for every $u \in E^+$.

Definition 2.5 Let $E$ be a Banach lattice. A linear bounded operator $T : E \to E$ is said to be unbounded order weakly demicompact if, for every order bounded sequence $(x_n)$ in $E^+$ such that $(x_n - Tx_n)$ unbounded norm converges to 0, and $(x_n)$ weakly converges to 0 as $n \to \infty$, we have $(x_n)$ unbounded norm convergent to 0 as $n \to \infty$, where $E^+$ is the positive cone of $E$.

An operator $T$ from $E$ into $E$ is called order weakly demicompact if, for every order bounded sequence $(x_n)$ in $E^+$ such that $x_n \to 0$ in $\sigma(E, E')$ and $||x_n - Tx_n|| \to 0$ as $n \to \infty$, we have $||x_n|| \to 0$ as $n \to \infty$, [3].

Definition 2.6 [5] Let $E$ be a Banach lattice. An operator $T : E \to E$ is said to be a demi KB-operator if, for every positive increasing sequence $(x_n)$ in the closed unit ball of $E$ such that $(x_n - Tx_n)$ is norm convergent to $x \in E$, there is a norm convergent subsequence of $(x_n)$.

If $\alpha \neq 1$, then $\alpha I$ is a demi KB-operator on a Banach lattice $E$. Every KB-operator on a Banach lattice is a demi KB-operator, [5].

Definition 2.7 [5] Let $E$ be a Banach lattice. An operator $T : E \to E$ is called a weak demi KB-operator if, for every positive increasing sequence $(x_n)$ in the closed unit ball of $E$ such that $(x_n - Tx_n)$ is weakly convergent to $x \in E$, there is a weakly convergent subsequence of $(x_n)$.
In [5], the authors proved that weak demi KB-operators and demi KB-operators on a Banach lattice are equivalent.

**Definition 2.8** An operator \( T : E \rightarrow E \) on a Banach lattice \( E \) is called unbounded norm demi KB-operator if for every positive increasing sequence \((x_n)\) in the closed unit ball of \( E \) such that \((x_n - Tx_n)\) is unbounded norm convergent to \( x \in E \), there is an unbounded norm convergent subsequence of \((x_n)\).

**Definition 2.9** Let \( E \) be a Banach lattice. An operator \( T : E \rightarrow E \) is said to be weakly unbounded demi KB-operator if, for every positive increasing sequence \((x_n)\) in the closed unit ball of \( E \) such that \((x_n - Tx_n)\) is weakly unbounded convergent to \( x \in E \), there is a weakly unbounded convergent subsequence of \((x_n)\).

**Theorem 2.10** Let \( E \) be a Banach lattice. Then, every KB-operator \( T : E \rightarrow E \) is an unbounded norm demi KB-operator.

Proof. Let \((x_n)\) be a positive increasing sequence in the closed unit ball of \( E \) such that \((x_n - Tx_n)\) is unbounded norm convergent to \( x \in E \). Since \( T \) is a KB-operator, there exists a subsequence \((Tx_{n(k)})\) of \((Tx_n)\) which is norm convergent to \( y \in E \). Hence, from the relation

\[
|x_{n(k)} - (x + y)| \wedge u = |(x_{n(k)} - Tx_{n(k)} + Tx_{n(k)} - (x + y))| \wedge u,
\]

for every \( u \in E^+ \).

\[
|x_{n(k)} - (x + y)| \wedge u = |(x_{n(k)} - Tx_{n(k)} - x) + (Tx_{n(k)} - y)| \wedge u,
\]

\[
\leq |(x_{n(k)} - Tx_{n(k)} - x)| + |(Tx_{n(k)} - y)| \wedge u,
\]

\[
\leq |(x_{n(k)} - Tx_{n(k)} - x)| \wedge u + |(Tx_{n(k)} - y)| \wedge u,
\]

for every \( u \in E^+ \). By taking norm both sides, we get

\[
\| |x_{n(k)} - (x + y)| \wedge u \| = \| |(x_{n(k)} - Tx_{n(k)} + Tx_{n(k)} - (x + y))| \wedge u \|
\]

\[
\leq \| |(x_{n(k)} - Tx_{n(k)} - x)| \wedge u \| + \| |(Tx_{n(k)} - y)| \wedge u \|
\]

for every \( u \in E^+ \). As \( n \rightarrow \infty \), the sequence \((x_{n(k)})\) is unbounded norm convergent to \( x + y \). Therefore, \( T \) is an unbounded demi KB-operator.
Theorem 2.11 Let $E$ be a Banach lattice. Every KB-operator $T : E \to E$ is a weakly unbounded demi KB-operator.

Theorem 2.12 Let $E$ be a Banach lattice. Let $T, S : E \to E$ be operators. If $T$ is an unbounded norm demi KB-operator and $S$ is a KB-operator, then $T + S$ is an unbounded norm demi KB-operator.

Proof. Let $(x_n)$ be a positive increasing sequence in the closed unit ball of $E$ such that $(x_n - (S + T)x_n)$ is unbounded norm convergent to $x \in E$. We have to show that $(x_n)$ contains an unbounded norm convergent subsequence. Since $S$ is a KB-operator, we infer that $(Sx_n)$ has a norm convergent subsequence $(Sx_{k(n)})$ which is norm convergent to $y \in E$. Since we can write

$$x_{k(n)} - Tx_{k(n)} = x_{k(n)} - (T + S)x_{k(n)} + Sx_{k(n)},$$

it follows that $(x_{k(n)} - Tx_{k(n)})$ is unbounded norm convergent to $x + y$. The fact that $T$ is unbounded norm demi KB-operator, we obtain that $(x_{k(n)})$ contains an unbounded norm convergent subsequence. Hence, $T + S$ is an unbounded norm demi KB-operator.

$$|x_{k(n)} - Tx_{k(n)} - (x + y)| \land u = |(x_{k(n)} - (T + S)x_{k(n)} - x) + (Sx_{k(n)} - y)| \land u \leq |(x_{k(n)} - (T + S)x_{k(n)} - x)| \land u + |(Sx_{k(n)} - y)| \land u$$

By taking norm both sides, we get

$$\|x_{k(n)} - Tx_{k(n)} - (x + y)| \land u\| \leq \|(x_{k(n)} - (T + S)x_{k(n)} - x)| \land u\| + \|(Sx_{k(n)} - y)| \land u\|$$

for every $u \in E^+$. As $n \to \infty$, we have $\|x_{k(n)} - Tx_{k(n)} - (x + y)| \land u\| \to 0$.

Theorem 2.13 Let $E$ be a Banach lattice. Consider the following statements.

1. $E$ is a KB-space.
2. Every operator $T : E \to E$ is a KB-operator.
3. Every operator $T : E \to E$ is an unbounded norm demi KB-operator.
4. The identity operator $I : E \to E$ is an unbounded norm demi KB-operator. Then, the implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ are true.

Definition 2.14 Let $E$ be a Banach lattice. $E$ is called unbounded norm KB-space if for every positive increasing sequence $(x_n)$ in the closed unit ball of $E$, $(x_n)$ is unbounded norm convergent.

Note that every KB-space is an unbounded norm KB-space.
**Definition 2.15** Let $E$ be a Banach lattice. An operator $T : E \to E$ is said to be an unbounded norm $KB$-operator if for every positive increasing sequence $(x_n)$ in the closed unit ball of $E$, the sequence $(Tx_n)$ is unbounded norm convergent in $E$.

Note that a Banach lattice $E$ is an unbounded norm $KB$ space if and only if the identity operator $I : E \to E$ is an unbounded norm $KB$-operator.

**Theorem 2.16** Let $E$ be a Banach lattice and $S, T : E \to E$ be operators satisfying $0 \leq S \leq T$. If $T$ is an unbounded norm $KB$-operator, then $S$ is an unbounded norm $KB$-operator.

**Theorem 2.17** Let $E$ be a Banach lattice. Let $S, T : E \to E$ be operators. If $T$ is an unbounded norm demi $KB$-operator and $S$ is an unbounded norm $KB$-operator, then $T + S$ is an unbounded norm demi $KB$-operator.

**Theorem 2.18** Let $E$ be a Banach lattice and $S, T : E \to E$ be operators satisfying $0 \leq S \leq T \leq I$. If $T$ is an unbounded norm demi $KB$-operator, then $S$ is an unbounded norm demi $KB$-operator.

Proof. Suppose that $S, T : E \to E$ are operators such that $0 \leq S \leq T \leq I$ and $T$ is an unbounded demi $KB$-operator. Let $(x_n)$ be a positive increasing sequence in the closed unit ball of $E$ such that $(x_n - Sx_n)$ is unbounded norm convergent to $x \in E$. We have to show that $(x_n)$ contains an unbounded norm convergent subsequence. By the hypothesis, $(I - S)$ is unbounded $KB$-operator. Since $0 \leq I - T \leq I - S$ holds and we obtain $I - T$ that is an unbounded $KB$-operator. The fact that $T$ is an unbounded norm demi $KB$-operator, we derive that $I = I - T + T$ is an unbounded norm demi $KB$-operator. So, $(x_n)$ has an unbounded norm convergent subsequence.

**Definition 2.19** An operator $T : X \to Y$ between Banach spaces is called a Dunford-Pettis operator if $(Tx_n)$ converges to zero in norm whenever $(x_n)$ converges weakly to zero.

**Theorem 2.20** Every Dunford-Pettis operator $T : E \to E$ on a Banach lattice $E$ is an unbounded norm demi $KB$-operator.

Proof. Every Dunford-Pettis operator is a $KB$-operator, [2]. So, it is an unbounded norm demi $KB$-operator.

For example, the identity operator $I : l^2 \to l^2$ is a $KB$-operator, and so it is an unbounded norm demi $KB$-operator. But, it is not a Dunford-Pettis operator.
Theorem 2.21 Let $E$ be a Banach lattice. Then, the following statements are equivalent.

1. $E$ is an unbounded norm $KB$-space.
2. Every operator $T : E \to E$ is an unbounded norm $KB$-operator.
3. Every operator $T : E \to E$ is an unbounded norm demi $KB$-operator.
4. The identity operator $I : E \to E$ is unbounded norm demi $KB$-operator.

Theorem 2.22 Every Dunford-Pettis operator $T : E \to E$ on a Banach lattice $E$ is an unbounded norm $KB$-operator.

Proof. Since every Dunford-Pettis operator is a $KB$-operator and every $KB$-operator is an unbounded norm $KB$-operator. So, it is an unbounded norm $KB$-operator.

Theorem 2.23 Let $E$ be a Banach lattice and $T : E \to E$ be a continuous linear operator. If the second adjoint $T'' : E'' \to E''$ of $T$ is order weakly compact, then $T$ is unbounded norm demi $KB$-operator.

Proof. Let $(x_n)$ be a positive increasing sequence in the closed unit ball of $E$. The set $B = \{x_n : n \in \mathbb{N} \}$ is an order bounded subset in $E''$. By the assumption $T''(B) = T(B)$ is a relatively weakly compact subset of $E$. So, $(Tx_n)$ has a weakly convergent subsequence. That is, $T$ is weakly $KB$-operator. and so $T$ is $KB$-operator, [2]. This implies that $T$ is an unbounded norm demi $KB$-operator.

Definition 2.24 A linear bounded operator $T : X \to E$ from a Banach space into a Banach lattice is called semicompact if for every $\varepsilon > 0$ there is an $x \in E^+$ such that $T(Ball(X)) \subseteq [-x,x] + \varepsilon Ball(E)$, where $Ball(X)$ and $Ball(E)$ denote closed unit balls of $X$ and $E$.

Theorem 2.25 Let $E$ be a Banach lattice and $T : E \to E$ be a bounded linear operator. If $T' : E' \to E'$ is semicompact, then $T$ is unbounded norm demi $KB$-operator.

Proof. Semicompactness of the adjoint operator $T'$ of $T$ implies that $T$ is a $KB$-operator. So, it is an unbounded norm demi $KB$-operator, [2].

References


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