On Unbounded Order Convergence of Operator Nets

Ebru Aydogan

Yildiz Technical University
Graduate School of Science and Engineering
Mathematics Department,
34220, Davutpasa, Istanbul, Turkey

Elif Demir

Yildiz Technical University
Faculty of Arts and Sciences
Mathematics Department
34220, Davutpasa, Istanbul, Turkey

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Abstract

In this study, we investigate uo-convergence of nets of positive continuous operators defined on the topological dual of a completely regular Hausdorff topological space. Firstly, we make the definition of uo-convergence on this class and then we present some characterizations of it.

Mathematics Subject Classification: 46A40, 47B38, 47B60

Keywords: uo-convergence, topological dual, completely regular Hausdorff space, nets of positive operators

1The author was granted by TUBITAK (The Scientific and Technological Research Council of Türkiye
1 Introduction

The notion of unbounded order convergence, simply uo-convergence, is firstly studied in the article [7] by a mathematician Hidegoro Nakano. The aim of Nakano was to define almost everywhere convergence in terms of lattice operations without direct use of measure theory and he defined "individual convergence". Then, it was named as "unboundedly order convergence" in the paper [3] by Ralph DeMaar. The development of this subject continued with [4], [5], [8], [9] and other researchers. We refer the reader to [2] for a survey of some convergence types on vector lattices.

In section 2, we give some basic definitions of Riesz spaces and some convergences on it. Let $X$ be a set with an order relation $\leq$. $X$ is said to be a lattice if any two elements $x, y \in X$ have a least upper bound and a greatest lower bound such that $x \vee y = \sup \{x, y\}$ and $x \wedge y = \inf \{x, y\}$. A lattice $(X, \leq)$ is called a vector lattice if the order and vector space structure are compossible. A net is a function from a directed set $A$ to an arbitrary set (a vector lattice) $X$. It is denoted by $(x_\alpha)_{\alpha \in A}$ or simply $(x_\alpha)$ if the index set is clear from a context. We refer the reader to [6] to for detailed information about vector lattices.

In section 3, we characterize uo-convergence of nets of operators which are defined on continuous functionals. In the paper [1], order and uo-convergence on $C(\Omega)$ were characterized where $\Omega$ is a Baire space and $C(\Omega)$ is the space of all real valued functions. We study the uo-convergence of operator nets defined on the space of continuous functionals satisfying the same conditions in [1].

2 Preliminaries

**Definition 2.1** Let $X$ be a vector lattice. The positive cone $X_+$ consists of all $x \in X$ such that $x \geq 0$. Furthermore, for every $x \in X$ let

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \quad \text{and} \quad |x| = x \vee (-x)$$

be the positive part, the negative part and the absolute value of $x$, respectively.

**Definition 2.2** Let $(x_\alpha)$ be a net indexed by a directed set $A$. For $\alpha_0 \in A$ fixed, let $A_0 = \{\alpha \in A : \alpha \geq \alpha_0\}$ which is again a directed set under the pre-order induced from $A$. The restriction of the function $x$ to $A_0$ is called a tail of $(x_\alpha)$, and it is denoted by $(x_\alpha)_{\alpha \geq \alpha_0}$.

**Definition 2.3** Let $(x_\alpha)_{\alpha \in A}$ be a net in $X$. $(x_\alpha)$ is said to be order convergent to $x$ if there exists a net $(y_\alpha)_{\alpha \in A}$ such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all $\alpha \in A$. It is denoted as $x_\alpha \rightarrow x$. 
Lemma 2.4 (Lemma 2.1, [1]) For a net \((x_\alpha)\) in a vector lattice \(X\), \(x_\alpha \xrightarrow{o} x\) if and only if there exists a set \(G \subseteq X_+\) such that \(\inf G = 0\) and every element of \(G\) dominates a tail of \((x_\alpha)\), that is, for every \(g \in G\) there exists \(\alpha_0\) such that \(|x_\alpha| \leq g\) for all \(\alpha \geq \alpha_0\).

Definition 2.5 A net \((x_\alpha)\) is said to be unbounded order converges to \(x\) if \(|x_\alpha - x| \wedge y \xrightarrow{o} 0\) for every \(y \geq 0\). We will use the notation \((x_\alpha)\) uo-converges to \(x\), for short, and we will denote it as \(x_\alpha \xrightarrow{uo} x\).

Order and uo-convergences agree for order bounded nets. If \(w \geq 0\) is a weak unit then \(x_\alpha \xrightarrow{uo} x\) if and only if \(|x_\alpha - x| \wedge w \xrightarrow{o} 0\) [1].

Definition 2.6 i. A subspace \(Y\) of \(X\) is called a sublattice of \(Y\) if \(x \vee y \in Y\) and \(x \wedge y \in Y\) for all \(x, y \in Y\).
ii. A sublattice \(Y\) of \(X\) is called order dense if \(0 < x \in X\) implies that there exists \(y \in Y\) with \(0 < y \leq x\).
iii. A sublattice is regular if the inclusion map is order continuous, that is, it preserves order convergence of nets.

Every order dense sublattice is regular. For a net \((x_\alpha)\) in a regular sublattice \(Y\) of \(X\), \(x_\alpha \xrightarrow{uo} 0\) in \(X\) if and only if \(x_\alpha \xrightarrow{uo} 0\) in \(Y\).

Definition 2.7 A net \((x_\alpha)_{\alpha \in A}\) is called order Cauchy (or simply o-Cauchy) if the double net \((x_\alpha - x_\beta)_{\alpha, \beta \in A}\) is order null. Unbounded order Cauchy (simply uo-Cauchy) is defined in the same way.

Before giving the Uryson’s lemma, recall that a normal space is a topological space that satisfies the Axiom \(T_4\), that is, every disjoint closed sets of a topological space have disjoint open neighborhoods. In addition, a normal Hausdorff space is called \(T_4\) space.

Theorem 2.8 (Uryson’s Lemma) A topological space \(X\) is normal if and only if for any two disjoint nonempty closed subsets \(Y, Z \subseteq X\) there is a continuous function \(f : X \to [0, 1]\) such that \(f(x) = 0\) for all \(x \in Y\) and \(f(x) = 1\) for all \(x \in Z\).

Definition 2.9 A topological space \(X\) is completely regular if the points can be seperated from closed sets via continuous real-valued functions. That is, for any closed set \(A \subseteq X\) and any point \(x \in X \setminus A\), there exists a real-valued continuous function \(f : X \to \mathbb{R}\) such that

\[ f(x) = 1 \]

and

\[ f|_A = 0. \]
Throughout the paper, $X$ stands for a completely regular Hausdorff topological space which is exactly the class of Hausdorff spaces where the conclusion of Uryson’s Lemma holds. As a consequence of Uryson’s Lemma, we can say that every locally compact Hausdorff space or every normal space is completely regular.

Let $X'$ be a topological dual of $X$ which we have mentioned above. $C(X)$ denotes the space of all real-valued continuous functions on $X$, and $1$ for the constant one function. Let $C(X')_+$ be a family of all continuous positive operators defined as follows:

$$C(X')_+ = \{ T : X' \rightarrow X' | Tf \geq 0 \text{ for } f \in X' \}. \quad (1)$$

3 Main Results

The following lemma essentially Lemma 3.1 in [1], we rewrite it for our main goal:

**Lemma 3.1** Suppose that $X$ is a completely regular Hausdorff topological space, $X'$ be the topological dual of it and $H \subseteq C(X')_+$. The following are equivalent:

i. $\inf H = 0$;

ii. for every non-empty open set $U$, $g \in X'$ and every $\varepsilon > 0$, there exists an element $s \in U$ and $T \in H$, with $(Tg)(s) < \varepsilon$;

iii. for every non-empty open set $U$, $g \in X'$ and every $\varepsilon > 0$, there exists a non-empty open set $V \subseteq U$ and $T \in H$ such that $(Tg)(s) < \varepsilon$ for all $s \in V$.

**Proof.** (i) $\Rightarrow$ (ii) Consider that $\inf H = 0$, but (ii) fails, that is, there exists an open non-empty $U$, $g \in X'$ and $\varepsilon > 0$ such that for every $T \in H$, $(Tg)(s) \geq \varepsilon$ on $U$ for all $s$. Since $X$ is completely regular, we find a nonzero $f \in C(X)_+$ and an operator $S \in C(X')_+$ such that $Sf \leq \varepsilon 1$ and $Sf$ vanishes outside of $U$. Then $S \leq H$, which contradicts $\inf H = 0$.

(ii) $\Rightarrow$ (i) Suppose that (ii) holds and $\inf H \neq 0$. Then there exists $S \in C(X')_+$ such that $0 < S \leq H$. Next, we can find an open non-empty set $U$, $f \in C(X)$ and $\varepsilon > 0$ such that $Sf > \varepsilon$ on $U$. It follows that every $T \in H, g \in X'$ we obtain that $Tg$ is greater than $\varepsilon$ on $U$, which contradicts (ii).

Other implications can be clearly seen. \qed

Now, we will define $uo$-convergence and characterize it on $C(X')_+$ by using operator nets.
**Definition 3.2** Let $T_\lambda$ be a net in $C(X')_+$. $T_\lambda \overset{\omega}{\rightarrow} T$ if and only if $|T_\lambda - T| \wedge f \overset{\omega}{\rightarrow} 0$ for all $f \in C(X')_+$. Since $T_\lambda \overset{\omega}{\rightarrow} T$ if and only if $|T_\lambda - T| \overset{\omega}{\rightarrow} 0$, it is enough to characterize $T_\lambda \overset{\omega}{\rightarrow} 0$.

**Theorem 3.3** Let $X'$ be a dual of a completely regular Hausdorff topological space $X$ and $(T_\lambda)$ be a net in $C(X')_+$. Then $T_\lambda \overset{\omega}{\rightarrow} 0$ if and only if for every non-empty open set $U$, $f \in X'_+$ and every $\varepsilon > 0$, there exists an open non-empty $V \subseteq U$ and an index $\lambda_0$ such that $T_\lambda f$ is less than $\varepsilon$ on $V$ whenever $\lambda \geq \lambda_0$.

**Proof.** Let $T_\lambda \overset{\omega}{\rightarrow} 0$. Then $T_\lambda \wedge 1 \overset{\omega}{\rightarrow} 0$. By the Lemma 2.4, we can find a set $H \subseteq C(X')_+$ such that $\inf H = 0$ and every element of $H$ dominates a tail of $(T_\lambda \wedge 1)$. Fix a non-empty open $U$ and $\varepsilon \in (0, 1)$. Choose such $V$ and $T$ that Lemma 3.1 (iii) holds. Since $T \in H$ then $T$ dominates a tail of $(T_\lambda \wedge 1)$, followingly, there exists $\lambda_0$ such that $T_\lambda \wedge 1 \leq T$ for all $\lambda \geq \lambda_0$. In particular,

$$(T_\lambda f)(s) \wedge 1(s) = (T_\lambda f)(s) \wedge 1 \leq (T f)(s) < \varepsilon$$

hence $(T_\lambda f)(s) < \varepsilon$ for all $s \in V$. This completes the proof.

Conversely, suppose that the condition in the theorem holds. Since $1$ is a weak unit, it is enough to prove that $T_\lambda \wedge 1 \overset{\omega}{\rightarrow} 0$. We will use Lemma 2.4 to prove it. Fix a nonempty set $U$ and $\varepsilon > 0$. We assume that $V \subseteq U$ is a non-empty set and $\lambda_0$ is an index such that $T_\lambda f$ is less than $\varepsilon$ on $V$ for all $f \in X'_+$ whenever $\lambda \geq \lambda_0$. Fix any $s \in V$. Since $X$ is completely regular, we can find $g \in C(X)_+$ such that $g(s) = 0$ and $g$ equals 1 outside of $V$. Put $k = g \vee \varepsilon 1$. Then

$$k(s) = g(s) \vee \varepsilon \cdot 1(s) = \varepsilon.$$ We claim that $(T_\lambda \wedge 1)(f) \leq k$ for every $\lambda \geq \lambda_0$. Indeed, if $m \in V$ then

$$(T_\lambda f)(m) < \varepsilon \leq k(s)$$

and if $m \notin V$ then

$$[(T_\lambda \wedge 1)(f)](m) \leq 1 = g(s) \leq k(s).$$

Repeat this process for every pair $(U, \varepsilon)$ where $U$ is open and non-empty and $\varepsilon > 0$. Let $H$ be the set of the resulting functions $k$. Each such $k$ dominates a tail of $(T_\lambda \wedge 1)$ Lemma 3.1 implies $\inf H = 0$, completes the proof. \(\square\)

**Corollary 3.4** Let $X$ be completely regular, $X'$ be a topological dual of it and $(T_\lambda)_{\lambda \in A}$ be a net in $C(X')_+$. Then $(T_\lambda)$ is $\omega$–Cauchy if and only if for every non-empty set $U$, $f \in X'_+$ and every $\varepsilon > 0$, there exists an open non-empty $V \subseteq U$ and an index $\lambda_0$ such that

$$|(T_\lambda f)(s) - (T_\beta f)(s)| < \varepsilon$$

for all $s \in V$ and $\lambda, \beta \geq \lambda_0$. 

Proof. \((\Rightarrow)\) Let \((T_\lambda)\) be an \(uo - Cauchy\) net. That is, for \(\lambda, \beta \in A\), the net \((T_\lambda - T_\beta)_{A \times A}\) is \(uo\)-null. By the Theorem 3.3 we get the result directly.

\((\Leftarrow)\) It can be easily seen by using Theorem 3.3 again. \(\square\)

Acknowledgements. The authors would like to thank the BAP (Scientific Research Projects) YTU and TUBITAK (The Scientific and Technological Research Council of Turkey) for their scientific supports.

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Received: May 12, 2023; Published: June 14, 2023