International Mathematical Forum, Vol. 17, 2022, no. 1, 1 - 10
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2022.912300

On the Generalized Ramanujan-Nagell Equation $x^2 + (4c)^m = (c + 1)^n$

Nobuhiro Terai

Division of Mathematical Sciences
Department of Integrated Science and Technology
Faculty of Science and Technology, Oita University
700 Dannoharu, Oita 870–1192, Japan

Saya Nakashiki

Division of Computer Science and Intelligent Systems
Graduate school of Engineering, Oita University
700 Dannoharu, Oita 870–1192, Japan

Yudai Suenaga

Division of Computer Science and Intelligent Systems
Graduate school of Engineering, Oita University
700 Dannoharu, Oita 870–1192, Japan

Abstract

Let c be a positive integer with $c \geq 2$. Then we conjecture that the equation $x^2 + (4c)^m = (c + 1)^n$ has only the positive integer solution $(x, m, n) = (c - 1, 1, 2)$ except for the cases $c = 5, 7, 309$. In this paper, we verify that this conjecture is true for several cases under some conditions on c.

Mathematics Subject Classification: 11D61

Keywords: Generalized Ramanujan-Nagell equation, Generalized Lebesgue-Ramanujan-Nagell equation, Zsigmondy's theorem, integer solution

1 Introduction

In 1913, Ramanujan [R] conjectured that the equation $x^2+7=2^n$ has only the positive integer solutions $(x,n)=(1,3),\ (3,4),\ (5,5),\ (11,7),\ (181,15)$. In 1960, Nagell [N] resolved Ramanujan's conjecture. Let b and c be fixed relatively prime positive integers greater than one. Then the generalized Ramanujan-Nagell equation

$$x^2+b^m=c^n$$

in positive integers x,m and n has been studied by a number of authors: (cf. [CD1], [CD2], [DGX], [Le1], [Le2], [Le3], [M], [To2] and [YW])

- (Tanahashi [Ta], Toyoizumi [To1]) $x^2+7^m=2^n$
- (Alter-Kubota [AK], Tanahashi [Ta]) $x^2+11^m=3^n$
- (Bugeaud[Bu]) $x^2+D^m=2^n$
- (Yaun-Hu[YH]]) $x^2+D^m=p^n$
- (Terai [Te1], [Te2]) $x^2+q^m=p^n, \quad x^2+q^m=c^n$

In the previous paper [Te2], the first author showed that if $2c-1$ is a prime and $2c-1\equiv 3,5 \pmod 8$, then the equation $x^2+(2c-1)^m=c^n$ has only the positive integer solution $(x,m,n)=(c-1,1,2)$, and proposed the following:

Conjecture 1. *Let c be a positive integer with $c\geq 2$. Then the equation*

$$x^2+(2c-1)^m=c^n$$

has only the positive integer solution $(x,m,n)=(c-1,1,2)$.

In [Te2], it was verified that if $2\leq c\leq 30$ with $c\neq 12,24$, then Conjecture 1 is true. The proof is based on elementary methods and a result concerning the Diophantine equation $\dfrac{x^n-1}{x-1}=y^2$ due to Ljunggren. Deng [D1] settled the cases $c=12,24$ by applying arithmetic properties of real quadratic fields $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{6})$, respectively. Fujita-Terai [FT] showed that if $2c-1=3p^l$ or $2c-1=5p^l$, then Conjecture 1 is true without any congruence condition on a prime p.

As an analogue of Conjecture 1, we propose the following:

Conjecture 2. *Let c be a positive integer with $c\geq 2$. Then the equation*

$$x^2+(4c)^m=(c+1)^n \tag{1.1}$$

has only the positive integer solution $(x, m, n) = (c-1, 1, 2)$ *except for the cases* $c = 5, 7, 309$, *where equation* (1.1) *has only the following positive integer solutions, respectively*:

$$\begin{aligned} c = 5; &\quad (x,m,n) = (4,1,2),\ (14,1,3), \\ c = 7; &\quad (x,m,n) = (6,1,2),\ (22,1,3),\ (104,3,5), \\ c = 309; &\quad (x,m,n) = (308,1,2),\ (5458,1,3). \end{aligned}$$

In this paper, we verify that this conjecture is true for several cases under some conditions on c. Our main result is the following:

Theorem 1. *Suppose that at least one of the following conditions is satisfied.*

(i) $c = 2^k$, *where* k *is a positive integer.*

(ii) $c = 2^k - 1$ $(k \geq 2)$.

(iii) $c = p^k - 1$, *where* p *is a prime with* $p \equiv 3 \pmod 4$.

(iv) $c = p^k$, *where* p *is a prime with* $p \equiv 3 \pmod 8$ *and* k *is odd.*

(v) $c = 2p^k$, *where* p *is a prime with* $p \equiv 1 \pmod 4$.

(vi) $c = 4p^k$, *where* p *is an odd prime with* $p^k \not\equiv 5 \pmod 8$.

Then Conjecture 2 *is true.*

The organization of this paper is as follows. In Section 2, we quote results on the generalized Lebesgue-Ramanujan-Nagell equations $x^2 + D^m = p^n$ with p prime and $x^2 \pm 2^m = y^n$, and Zsigmondy's theorem concerning primitive prime divisor. In Section 3, by elementary methods, we solve the exponential Diophantine equations $2^l + q^m = (2^\alpha q + 1)^n$ and $2^l q^m + 1 = (2^\alpha q + 1)^n$ with $\alpha = 1, 2$ under some conditions. In Section 4, we use Propositions and Lemmas in Sections 2,3 to show Theorem 1.

2 Preliminaries

In the proof of Theorem 1, we need the following five Propositions concerning the generalized Ramanujan-Nagell equations, Lebesgue-Ramanujan-Nagell equations and the Primitive Divisor Theorem due to Zsigmondy:

Proposition 1 (Bugeaud[Bu])**.** *Let D be an odd positive integer. Then the equation*

$$x^2 + D^m = 2^n$$

in positive integers x, m, n has at most one solution (x, m, n), except for the cases $D = 7,\ 23,\ 2^k - 1\ (k \geq 4)$, where the equation has only the following solutions, respectively.

(i) $c = 7$; $(x, m, n) = (1, 1, 3), (3, 1, 4), (5, 1, 5), (11, 1, 7), (181, 1, 15), (13, 3, 9)$.
(ii) $c = 23$; $(x, m, n) = (3, 1, 5), (45, 1, 11)$.
(iii) $c = 2^k - 1\ (k \geq 4)$; $(x, m, n) = (308, 1, 2),\ (5458, 1, 3)$.

Remark 1. In Theorem 3 of Bugeaud[Bu], it was stated that the exceptional cases are $D = 7,\ 15$. But we point out that the ones are $D = 7,\ 23,\ 2^k - 1\ (k \geq 4)$. (cf. Theorem 2 of Beukers[Be].)

Proposition 2 (Bugeaud[Bu], Yaun-Hu[YH])**.** *Let $D > 2$ be an integer and let p be an odd prime not dividing D. If $(D, p) \neq (4, 5)$, then the equation*

$$x^2 + D^m = p^n$$

has at most two positive integer solutions (x, m, n). If the two solutions are (x_1, m_1, n_1) and (x_2, m_2, n_2), then $m_1 \not\equiv m_2 \pmod 2$. The equation $x^2 + 4^m = 5^n$ has exactly three positive integer solutions (x, m, n).

Proposition 3 (Le[Le4])**.** *Then the equation*

$$x^2 + 2^m = y^n, \quad \gcd(x, y) = 1,\ n \geq 3$$

has only the positive integer solutions $(x, y, m, n) = (5, 3, 1, 3), (7, 3, 5, 4), (11, 5, 2, 3)$.

Proposition 4 (Ivorra[I])**.** *The equation*

$$x^2 - 2^m = y^n, \quad \gcd(x, y) = 1,\ |y| > 1,\ m \geq 2,\ n \geq 3$$

has only the integer solutions $(x, y, m, n) = (\pm 13, -7, 9, 3),\ (\pm 71, 17, 7, 3)$.

Proposition 5 (Zsigmondy [Z])**.** *Let A and B be relatively prime integers with $A > B \geq 1$. Let $\{a_k\}_{k \geq 1}$ be the sequence defined as*

$$a_k = A^k + B^k.$$

If $k > 1$, then a_k has a prime factor not dividing $a_1 a_2 \cdots a_{k-1}$, whenever $(A, B, k) \neq (2, 1, 3)$.

3 the exponential Diophantine equations

We use the following Lemmas 1, 2 to show Theorem 1 (v),(vi), respectively.

Lemma 1. *Let q be an odd integer with $q \geq 3$.*

(1) *If $q \equiv 1 \pmod 4$, then the equation*

$$2^{3m-2} + q^m = (2q+1)^n \tag{3.1}$$

has no positive integer solutions (m, n).

(2) *If $q \equiv 1 \pmod 4$, then the equation*

$$2^{3m-2}q^m + 1 = (2q+1)^n \tag{3.2}$$

has only the positive integer solution $(m, n) = (1, 1)$.

Proof. (1) It is clear that if $m = 1$, then equation (3.1) has no solutions. We may thus suppose that $m > 1$. Taking (3.1) modulo 4 implies that $q^m \equiv 3^n \pmod 4$. In view of $q \equiv 1 \pmod 4$, we see that n is even. Then it follows from Proposition 4 that equation (3.1) has no solutions.

(2) If $m = 1$, then equation (3.2) has only the solution $n = 1$. We may thus suppose that $m > 1$. Then taking (3.2) modulo 4 implies that $1 \equiv 3^n \pmod 4$. Hence n is even, say $n = 2N$. Then

$$2^{3m-2}q^m = ((2q+1)^2 - 1)\frac{(2q+1)^{2N} - 1}{(2q+1)^2 - 1} = 2q \cdot (2q+2)\frac{(2q+1)^{2N} - 1}{(2q+1)^2 - 1}.$$

Since $\gcd(q+1, q) = 1$, the above implies that $(q+1) | 2^{3m-2}$, which is impossible, since $q \equiv 1 \pmod 4$. □

Lemma 2. *Let q be an odd integer with $q \geq 3$.*

(1) *If $q \not\equiv 5 \pmod 8$, then the equation*

$$2^{4m-2} + q^m = (4q+1)^n \tag{3.3}$$

has no positive integer solutions (m, n).

(2) *The equation*

$$2^{4m-2}q^m + 1 = (4q+1)^n \tag{3.4}$$

has only the positive integer solution $(m, n) = (1, 1)$.

Proof. (1) It is clear that if $m = 1$, then equation (3.3) has no solutions. We may thus suppose that $m > 1$. Taking (3.3) modulo 8 implies that $q^m \equiv 5^n \pmod 8$. In view of $q \not\equiv 5 \pmod 8$, we see that n is even. Then it follows from Proposition 4 that equation (3.1) has no solutions.

(2) If $m = 1$, then equation (3.4) has only the solution $n = 1$. We may thus suppose that $m > 1$. Then taking (3.4) modulo 8 implies that $1 \equiv 5^n \pmod 8$. Hence n is even, say $n = 2N$. Then

$$2^{4m-2}q^m = ((4q+1)^2 - 1)\frac{(4q+1)^{2N} - 1}{(4q+1)^2 - 1} = 4q \cdot (4q+2)\frac{(4q+1)^{2N} - 1}{(4q+1)^2 - 1}.$$

Since $\gcd(2q+1, q) = 1$, the above implies that $(2q+1)|2^{4m-2}$, which is impossible. □

4 Proof of Theorem 1

(i) Our assertion follows from Proposition 3.

(ii) Let (x, m, n) be a solution of equation (1.1). Suppse that our assumptions are all satisfied.

We first note that that $n > m$ from (1.1). Indeed,

$$(c+1)^n = x^2 + (4c)^m > (4c)^m > (c+1)^m.$$

Since x is even, we put $x = 2^\alpha x_1$ with $\alpha \geq 1$ and x_1 odd. Then equation (1.1) leads to

$$2^{2\alpha}x_1^2 + 2^{2m}c^m = 2^{kn}. \tag{4.1}$$

We want to show that $\alpha = m$. If $\alpha > m$, then equation (4.1) implies that

$$2^{2m}(2^{2\alpha-2m}x_1^2 + c^m) = 2^{kn},$$

so $2m = kn > 2m$ from $k \geq 2$ and $n > m$, which is impossible. If $\alpha < m$, then equation (4.1), as above, implies that $2\alpha = kn$, so $2m > 2\alpha = kn > 2m$, which is impossible. Consequently we obtain $\alpha = m$. Dividing both sides of (4.1) by 2^{2m} yields

$$x_1^2 + (2^k - 1)^m = 2^{kn-2m}.$$

Then our assertion easily follows from Proposition 1.

(iii) In view of $p \equiv 3 \pmod 4$, we see that m is odd. Then our assertion follows from Proposition 2.

(iv) Let (x, m, n) be a solution of equation (1.1). Suppse that our assumptions are all satisfied.

Put $c = p^k$ with $p \equiv 3 \pmod 8$ and k odd. Since $c \equiv 3 \pmod 8$, we can put $c+1 = 2^2 d$ with d odd. From equation (1.1), x is even, say $x = 2^\alpha x_1$ with $\alpha \geq 1$ and x_1 odd. Then equation (1.1) leads to

$$2^{2\alpha} x_1^2 + 2^{2m} c^m = 2^{2n} d^n. \tag{4.2}$$

Note that $n > m$ as before. We want to show that $\alpha = m$. If $\alpha > m$, then equation (4.3) implies that $n = m$, which contradicts the fact that $n > m$. If $\alpha < m$, then equation (4.3) implies that $n = \alpha < m$, which contradicts the fact that $n > m$. Hence we obtain $\alpha = m$, so

$$x_1^2 + c^m = 2^{2(n-m)} d^n. \tag{4.3}$$

Then it follows that $n - m = 1$, since $x_1^2 + c^m \equiv 1 + 3^m \not\equiv 0 \pmod 8$. From (4.3), we see that $1 + 3^m \equiv 4 \pmod 8$, so m is odd. Therefore equation (4.3) can be written as

$$c^m = \left(2d^{\frac{m+1}{2}} + x_1\right)\left(2d^{\frac{m+1}{2}} - x_1\right).$$

Since two factors of the right hand side of the above are relatively prime and $c = p^k$, we obtain the following:

$$\begin{cases} 2d^{\frac{m+1}{2}} + x_1 &= c^m \\ 2d^{\frac{m+1}{2}} - x_1 &= 1. \end{cases}$$

Adding these two equations yields

$$c^m + 1 = 4d^{\frac{m+1}{2}}. \tag{4.4}$$

From definition of d, we have

$$c + 1 = 4d.$$

If $m > 1$, then it follows from Proposition 5 that equation (4.4) has no solutions. Consequently we obtain $m = 1$, $n = 2$ and $x = c - 1$.

(v) Let (x, m, n) be a solution of equation (1.1). Suppse that our assumptions are all satisfied.

Put $q = p^k$ with $p \equiv 1 \pmod 4$and $C = 2q+1$. Then taking equation (1.1) modulo 4 implies that $1 \equiv 3^n \pmod 4$, so n is even, say $n = 2N$. From (1.1), we have

$$(2^3 q)^m = (C^N + x)(C^N - x).$$

Since $\gcd(C^N + x, C^N - x) = 2$ and $q = p^k$, we obtain the following two cases:

$$\begin{cases} C^N \pm x &= 2^{3m-1} \\ C^N \mp x &= 2q^m \end{cases} \tag{4.5}$$

or

$$\begin{cases} C^N \pm x = 2^{3m-1}q^m \\ C^N \mp x = 2. \end{cases} \tag{4.6}$$

First consider case (4.5). Adding these two equations yields

$$2^{3m-2} + q^m = (2q+1)^N,$$

which has no solutions by Lemma 1, (1).

Next consider case (4.6). Adding these two equations yields

$$2^{3m-2}q^m + 1 = (2q+1)^N,$$

which has only the solution $(m, N) = (1, 1)$ by Lemma 1, (2). Hence equation (1.1) has only the solution $(x, m, n) = (c-1, 1, 2)$.

(vi) Let (x, m, n) be a solution of equation (1.1). Suppse that our assumptions are all satisfied.

Put $q = p^k$ with $p^k \not\equiv 5 \pmod 8$ and $C = 4q + 1$. Then taking equation (1.1) modulo 8 implies that $1 \equiv 5^n \pmod 8$, so n is even, say $n = 2N$. From (1.1), we have

$$(2^4q)^m = (C^N + x)(C^N - x).$$

Since $\gcd(C^N + x, C^N - x) = 2$ and $q = p^k$, we obtain the following two cases:

$$\begin{cases} C^N \pm x = 2^{4m-1} \\ C^N \mp x = 2q^m \end{cases} \tag{4.7}$$

or

$$\begin{cases} C^N \pm x = 2^{4m-1}q^m \\ C^N \mp x = 2. \end{cases} \tag{4.8}$$

First consider case (4.7). Adding these two equations yields

$$2^{4m-2} + q^m = (4q+1)^N,$$

which has no solutions by Lemma 2, (1).

Next consider case (4.8). Adding these two equations yields

$$2^{4m-2}q^m + 1 = (4q+1)^N,$$

which has only the solution $(m, N) = (1, 1)$ by Lemma 2, (2). Hence equation (1.1) has only the solution $(x, m, n) = (c-1, 1, 2)$. This completes the proof of Theorem 1.

Acknowledgements The first author is supported by JSPS KAKENHI Grant Number 18K03247.

References

[AK] R. Alter and K.K. Kubota, The Diophantine Equation $x^2 + 11 = 3^n$ and a Related Sequence, *J. Number Theory*, **7** (1975), 5–10. https://doi.org/10.1016/0022-314x(75)90003-7

[Be] F. Beukers, On the generalize Ramanujan-Nagell equation I, *Acta Arith.*, **38** (1981), 389–410. https://doi.org/10.4064/aa-38-4-389-410

[BC] W. Bosma and J. Cannon, *Handbook of Magma Functions*, Department of Math., University of Sydney. http://magma.maths.usyd.edu.au/magma/

[Bu] Y. Bugeaud, On some exponential diophantine equations, *Monatsh. Math.*, **132** (2001), 93–97. https://doi.org/10.1007/s006050170046

[CD1] Z. Cao and X. Dong, On Terai's conjecture, *Proc. Japan Acad.*, **74A** (1998), 127–129.

[CD2] Z. Cao and X. Dong, The diophantine equation $x^2 + b^y = c^z$, *Proc. Japan Acad.*, **77A** (2001), 1-4.

[D1] M. Deng, A note on the Diophantine equation $x^2 + q^m = c^{2n}$, *Proc. Japan Acad., Series A, Mathematical Sciences*, **91** (2015), 15–18. https://doi.org/10.3792/pjaa.91.15

[DGX] M.Deng, J. Guo and A. Xu, A note on the Diophantine equation $x^2 + (2c-1)^m = c^n$, *Bull. Australian Math. Soc.*, **98** (2018), 188-195.

[FT] Y. Fujita and N. Terai, On the generalized Ramanujan-Nagell equation $x^2 + (2c-1)^m = c^n$, *Acta Math. Hungar.*, **162** (2020), 518–526. https://doi.org/10.1007/s10474-020-01085-8

[I] W. Ivorra, Sur les équations $x^p + 2^\beta y^p = z^2$ et $x^p + 2^\beta y^p = 2z^2$, *Acta Arith.*, **108** (2003) 327–338.

[Le1] M. Le, The diophantine equation $x^2 + D^m = p^n$, *Acta Arith.*, **52** (1989), 255-265.

[Le2] M. Le, A note on the diophantine equation $x^2 + b^y = c^z$, *Acta Arith.*, **71** (1995), 253–257.

[Le3] M. Le, On Terai's conjecture concerning Pythagorean numbers, *Acta Arith.*, **100** (2001), 41–45. https://doi.org/10.4064/aa100-1-3

[Le4] M. Le, On Cohn's conjecture concerning the Diophantine equation $x^2 + 2^m = y^n$, *Arch. Math.*, **78** (2002), 26–35. https://doi.org/10.1007/s00013-002-8213-5

[Lj] W. Ljunggren, Some theorems on indeterminate equations of the form $\frac{x^n - 1}{x - 1} = y^q$ (Norvegian), *Norsk Mat. Tidsskr.*, **25** (1943), 17–20.

[M] T. Miyazaki, A polynomial-exponential equation related to the Ramanujan-Nagell equation, *Ramanujan J.*, **45** (2018), 601-613. https://doi.org/10.1007/s11139-016-9878-x

[N] T. Nagell, The Diophantine equation $x^2 + 7 = 2^n$, *Ark. Math.*, **4** (1960), 185-187.

[R] S. Ramanujan, Question 446, *J. Indian Math. Soc.*, **5** (1913), 120. Collected papers, Cambridge University Press (1927), 327.

[Ta] K. Tanahashi, On the Diophantine equations $x^2+7^m = 2^n$ and $x^2+11^m = 3^n$, *J. Predent. Fac., Gifu Coll. Dent.*, **3** (1977), 77-79.

[Te1] N. Terai, The Diophantine Equation $x^2 + q^m = p^n$, *Acta Arith.*, **63** (1993), 351–358.

[Te2] N. Terai, A note on the Diophantine equation $x^2 + q^m = c^n$, *Bull. Australian Math. Soc.*, **90** (2014), 20–27.

[To1] M. Toyoizumi, On the diophantine equation $y^2+D^m = 2^n$, *Commentarii mathematici Universitatis Sancti Pauli*, **27** (1979), 105–111.

[To2] M. Toyoizumi, On the diophantine equation $x^2 + D^m = p^n$, *Acta Arith.*, **42** (1983), 303–309.

[YH] P. Yuan and Y. Hu, On the diophantine equation $x^2 + D^m = p^n$, *J. Number Theory*, **111** (2005), 144–153.

[YW] P. Yuan and J.B. Wang, On the diophantine equation $x^2 + b^y = c^z$, *Acta Arith.*, **84** (1998), 145–147.

[Z] K. Zsigmondy, Zur Theorie der Potenzreste, *Monatsh. Math.*, **3** (1892), 265–284.

Received: December 23, 2021; Published: January 16, 2022