On the Generalized Ramanujan-Nagell Equation

\[ x^2 + (4c)^m = (c + 1)^n \]

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Abstract

Let \( c \) be a positive integer with \( c \geq 2 \). Then we conjecture that the equation \( x^2 + (4c)^m = (c + 1)^n \) has only the positive integer solution \( (x, m, n) = (c - 1, 1, 2) \) except for the cases \( c = 5, 7, 309 \). In this paper, we verify that this conjecture is true for several cases under some conditions on \( c \).

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1 Introduction

In 1913, Ramanujan [R] conjectured that the equation $x^2 + 7 = 2^n$ has only the positive integer solutions $(x, n) = (1, 3), (3, 4), (5, 5), (11, 7), (181, 15)$. In 1960, Nagell [N] resolved Ramanujan’s conjecture. Let $b$ and $c$ be fixed relatively prime positive integers greater than one. Then the generalized Ramanujan-Nagell equation

$$x^2 + bm = cn$$

in positive integers $x, m$ and $n$ has been studied by a number of authors: (cf. [CD1], [CD2], [DGX], [Le1], [Le2], [Le3], [M], [To2] and [YW])

- (Tanahashi [Ta], Toyoizumi [To1]) $x^2 + 7m = 2^n$
- (Alter-Kubota [AK], Tanahashi [Ta]) $x^2 + 11m = 3^n$
- (Bugeaud [Bu]) $x^2 + Dm = 2^n$
- (Yaun-Hu [YH]) $x^2 + Dm = p^n$
- (Terai [Te1], [Te2]) $x^2 + qm = p^n$, $x^2 + qm = c^n$

In the previous paper [Te2], the first author showed that if $2c - 1$ is a prime and $2c - 1 \equiv 3, 5 \pmod{8}$, then the equation $x^2 + (2c - 1)m = c^n$ has only the positive integer solution $(x, m, n) = (c - 1, 1, 2)$, and proposed the following:

Conjecture 1. Let $c$ be a positive integer with $c \geq 2$. Then the equation

$$x^2 + (2c - 1)m = c^n$$

has only the positive integer solution $(x, m, n) = (c - 1, 1, 2)$.

In [Te2], it was verified that if $2 \leq c \leq 30$ with $c \neq 12, 24$, then Conjecture 1 is true. The proof is based on elementary methods and a result concerning the Diophantine equation $x^n - 1 = y^2$ due to Ljunggren. Deng [D1] settled the cases $c = 12, 24$ by applying arithmetic properties of real quadratic fields $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{6})$, respectively. Fujita-Terai [FT] showed that if $2c - 1 = 3p^l$ or $2c - 1 = 5p^l$, then Conjecture 1 is true without any congruence condition on a prime $p$.

As an analogue of Conjecture 1, we propose the following:

Conjecture 2. Let $c$ be a positive integer with $c \geq 2$. Then the equation

$$x^2 + (4c)^m = (c + 1)^n$$

(1.1)
On the generalized Ramanujan-Nagell equation $x^2 + (4c)^m = (c + 1)^n$ has only the positive integer solution $(x, m, n) = (c - 1, 1, 2)$ except for the cases $c = 5, 7, 309$, where equation (1.1) has only the following positive integer solutions, respectively:

- $c = 5; \quad (x, m, n) = (4, 1, 2), (14, 1, 3)$,
- $c = 7; \quad (x, m, n) = (6, 1, 2), (22, 1, 3), (104, 3, 5)$,
- $c = 309; \quad (x, m, n) = (308, 1, 2), (5458, 1, 3)$.

In this paper, we verify that this conjecture is true for several cases under some conditions on $c$. Our main result is the following:

**Theorem 1.** Suppose that at least one of the following conditions is satisfied.

(i) $c = 2^k$, where $k$ is a positive integer.

(ii) $c = 2^k - 1$ ($k \geq 2$).

(iii) $c = p^k - 1$, where $p$ is a prime with $p \equiv 3 \pmod{4}$.

(iv) $c = p^k$, where $p$ is a prime with $p \equiv 3 \pmod{8}$ and $k$ is odd.

(v) $c = 2p^k$, where $p$ is a prime with $p \equiv 1 \pmod{4}$.

(vi) $c = 4p^k$, where $p$ is an odd prime with $p^k \not\equiv 5 \pmod{8}$.

Then Conjecture 2 is true.

The organization of this paper is as follows. In Section 2, we quote results on the generalized Lebesgue-Ramanujan-Nagell equations $x^2 + D^m = p^n$ with $p$ prime and $x^2 \pm 2^m = y^n$, and Zsigmondy’s theorem concerning primitive prime divisor. In Section 3, by elementary methods, we solve the exponential Diophantine equations $2^l + q^m = (2^\alpha q + 1)^n$ and $2^l q^m + 1 = (2^\alpha q + 1)^n$ with $\alpha = 1, 2$ under some conditions. In Section 4, we use Propositions and Lemmas in Sections 2, 3 to show Theorem 1.

## 2 Preliminaries

In the proof of Theorem 1, we need the following five Propositions concerning the generalized Ramanujan-Nagell equations, Lebesgue-Ramanujan-Nagell equations and the Primitive Divisor Theorem due to Zsigmondy:
Proposition 1 (Bugeaud[Bu]). Let $D$ be an odd positive integer. Then the equation
\[ x^2 + D^m = 2^n \]
in positive integers $x, m, n$ has at most one solution $(x, m, n)$, except for the cases $D = 7$, $23$, $2^k - 1$ ($k \geq 4$), where the equation has only the following solutions, respectively.

(i) $c = 7$: $(x, m, n) = (1, 1, 3), (3, 1, 4), (5, 1, 5), (11, 1, 7), (181, 1, 15), (13, 3, 9)$.
(ii) $c = 23$: $(x, m, n) = (3, 1, 5), (45, 1, 11)$.
(iii) $c = 2^k - 1$ ($k \geq 4$): $(x, m, n) = (308, 1, 2), (5458, 1, 3)$.

Remark 1. In Theorem 3 of Bugeaud[Bu], it was stated that the exceptional cases are $D = 7, 15$. But we point out that the ones are $D = 7, 23, 2^k - 1$ ($k \geq 4$). (cf. Theorem 2 of Beukers[Be].)

Proposition 2 (Bugeaud[Bu], Yaun-Hu[YH]). Let $D > 2$ be an integer and let $p$ be an odd prime not dividing $D$. If $(D, p) \neq (4, 5)$, then the equation
\[ x^2 + D^m = p^n \]
has at most two positive integer solutions $(x, m, n)$. If the two solutions are $(x_1, m_1, n_1)$ and $(x_2, m_2, n_2)$, then $m_1 \equiv m_2$ (mod 2). The equation $x^2 + 4^m = 5^n$ has exactly three positive integer solutions $(x, m, n)$.

Proposition 3 (Le[Le4]). Then the equation
\[ x^2 + 2^m = y^n, \quad \gcd(x, y) = 1, \quad n \geq 3 \]
has only the positive integer solutions $(x, y, m, n) = (5, 3, 1, 3), (7, 3, 5, 4), (11, 5, 2, 3)$.

Proposition 4 (Ivorra[I]). The equation
\[ x^2 - 2^m = y^n, \quad \gcd(x, y) = 1, \quad |y| > 1, \quad m \geq 2, \quad n \geq 3 \]
has only the integer solutions $(x, y, m, n) = (\pm 13, -7, 9, 3), (\pm 71, 17, 7, 3)$.

Proposition 5 (Zsigmondy [Z]). Let $A$ and $B$ be relatively prime integers with $A > B \geq 1$. Let $\{a_k\}_{k \geq 1}$ be the sequence defined as
\[ a_k = A^k + B^k. \]
If $k > 1$, then $a_k$ has a prime factor not dividing $a_1a_2\cdots a_{k-1}$, whenever $(A, B, k) \neq (2, 1, 3)$. 
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3  the exponential Diophantine equations

We use the following Lemmas 1, 2 to show Theorem 1 (v), (vi), respectively.

**Lemma 1.** Let \( q \) be an odd integer with \( q \geq 3 \).

(1) If \( q \equiv 1 \pmod{4} \), then the equation

\[
2^{3m-2} + q^m = (2q + 1)^n \tag{3.1}
\]

has no positive integer solutions \((m, n)\).

(2) If \( q \equiv 1 \pmod{4} \), then the equation

\[
2^{3m-2}q^m + 1 = (2q + 1)^n \tag{3.2}
\]

has only the positive integer solution \((m, n) = (1, 1)\).

**Proof.** (1) It is clear that if \( m = 1 \), then equation (3.1) has no solutions. We may thus suppose that \( m > 1 \). Taking (3.1) modulo 4 implies that \( q^m \equiv 3^n \pmod{4} \). In view of \( q \equiv 1 \pmod{4} \), we see that \( n \) is even. Then it follows from Proposition 4 that equation (3.1) has no solutions.

(2) If \( m = 1 \), then equation (3.2) has only the solution \( n = 1 \). We may thus suppose that \( m > 1 \). Then taking (3.2) modulo 4 implies that \( 1 \equiv 3^n \pmod{4} \). Hence \( n \) is even, say \( n = 2N \). Then

\[
2^{3m-2}q^m = ((2q + 1)^2 - 1) \frac{(2q + 1)^{2N} - 1}{(2q + 1)^2 - 1} = 2q \cdot (2q + 2) \frac{(2q + 1)^{2N} - 1}{(2q + 1)^2 - 1}.
\]

Since \( \gcd(q+1, q) = 1 \), the above implies that \( (q+1)|2^{3m-2} \), which is impossible, since \( q \equiv 1 \pmod{4} \).

**Lemma 2.** Let \( q \) be an odd integer with \( q \geq 3 \).

(1) If \( q \not\equiv 5 \pmod{8} \), then the equation

\[
2^{4m-2} + q^m = (4q + 1)^n \tag{3.3}
\]

has no positive integer solutions \((m, n)\).

(2) The equation

\[
2^{4m-2}q^m + 1 = (4q + 1)^n \tag{3.4}
\]

has only the positive integer solution \((m, n) = (1, 1)\).
Proof. (1) It is clear that if \( m = 1 \), then equation (3.3) has no solutions. We may thus suppose that \( m > 1 \). Taking (3.3) modulo 8 implies that \( q^m \equiv 5^n \pmod{8} \). In view of \( q \not\equiv 5 \pmod{8} \), we see that \( n \) is even. Then it follows from Proposition 4 that equation (3.1) has no solutions.

(2) If \( m = 1 \), then equation (3.4) has only the solution \( n = 1 \). We may thus suppose that \( m > 1 \). Then taking (3.4) modulo 8 implies that \( 1 \equiv 5^n \pmod{8} \). Hence \( n \) is even, say \( n = 2N \). Then

\[
2^{4m-2}q^m = ((4q + 1)^2 - 1) \frac{(4q + 1)^{2N} - 1}{(4q + 1)^2 - 1} = 4q \cdot (4q + 2) \frac{(4q + 1)^{2N} - 1}{(4q + 1)^2 - 1}.
\]

Since \( \gcd(2q + 1, q) = 1 \), the above implies that \( (2q + 1)|2^{4m-2} \), which is impossible.

4 Proof of Theorem 1

(i) Our assertion follows from Proposition 3.

(ii) Let \((x, m, n)\) be a solution of equation (1.1). Suppose that our assumptions are all satisfied.

We first note that that \( n > m \) from (1.1). Indeed,

\[(c + 1)^n = x^2 + (4c)^m > (4c)^m > (c + 1)^m.\]

Since \( x \) is even, we put \( x = 2^\alpha x_1 \) with \( \alpha \geq 1 \) and \( x_1 \) odd. Then equation (1.1) leads to

\[2^{2\alpha} x_1^2 + 2^{2m} c^m = 2^{kn}. \tag{4.1}\]

We want to show that \( \alpha = m \). If \( \alpha > m \), then equation (4.1) implies that

\[2^{2m}(2^{2\alpha-2m} x_1^2 + c^m) = 2^{kn}, \]

so \( 2m = kn > 2m \) from \( k \geq 2 \) and \( n > m \), which is impossible. If \( \alpha < m \), then equation (4.1), as above, implies that \( 2\alpha = kn \), so \( 2m > 2\alpha = kn > 2m \), which is impossible. Consequently we obtain \( \alpha = m \). Dividing both sides of (4.1) by \( 2^{2m} \) yields

\[x_1^2 + (2^k - 1)^m = 2^{kn-2m}.\]

Then our assertion easily follows from Proposition 1.

(iii) In view of \( p \equiv 3 \pmod{4} \), we see that \( m \) is odd. Then our assertion follows from Proposition 2.

(iv) Let \((x, m, n)\) be a solution of equation (1.1). Suppose that our assumptions are all satisfied.
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Put $c = p^k$ with $p \equiv 3 \pmod{8}$ and $k$ odd. Since $c \equiv 3 \pmod{8}$, we can put $c + 1 = 2^d d$ with $d$ odd. From equation (1.1), $x$ is even, say $x = 2^\alpha x_1$ with $\alpha \geq 1$ and $x_1$ odd. Then equation (1.1) leads to

$$2^{2\alpha} x_1^2 + 2^{2m} c^m = 2^{2n} d^m. \tag{4.2}$$

Note that $n > m$ as before. We want to show that $\alpha = m$. If $\alpha > m$, then equation (4.3) implies that $n = m$, which contradicts the fact that $n > m$. If $\alpha < m$, then equation (4.3) implies that $n = \alpha < m$, which contradicts the fact that $n > m$. Hence we obtain $\alpha = m$, so

$$x^2 + c^m = 2^{2(n-m)} d^m. \tag{4.3}$$

Then it follows that $n - m = 1$, since $x_1^2 + c^m \equiv 1 + 3^m \not\equiv 0 \pmod{8}$. From (4.3), we see that $1 + 3^m \equiv 4 \pmod{8}$, so $m$ is odd. Therefore equation (4.3) can be written as

$$c^m = \left(2d^{m+1} + x_1\right)\left(2d^{m+1} - x_1\right).$$

Since two factors of the right hand side of the above are relatively prime and $c = p^k$, we obtain the following:

$$\begin{cases} 2d^{m+1} + x_1 = c^m \\ 2d^{m+1} - x_1 = 1. \end{cases} \tag{4.5}$$

Adding these two equations yields

$$c^m + 1 = 4d^{m+1}. \tag{4.4}$$

From definition of $d$, we have

$$c + 1 = 4d.$$

If $m > 1$, then it follows from Proposition 5 that equation (4.4) has no solutions. Consequently we obtain $m = 1$, $n = 2$ and $x = c - 1$.

(v) Let $(x, m, n)$ be a solution of equation (1.1). Suppose that our assumptions are all satisfied.

Put $q = p^k$ with $p \equiv 1 \pmod{4}$ and $C = 2q + 1$. Then taking equation (1.1) modulo 4 implies that $1 \equiv 3^n \pmod{4}$, so $n$ is even, say $n = 2N$. From (1.1), we have

$$(2^3 q)^m = (C^N + x)(C^N - x).$$

Since $\gcd(C^N + x, C^N - x) = 2$ and $q = p^k$, we obtain the following two cases:

$$\begin{cases} C^N \pm x = 2^{3m-1} \\ C^N \mp x = 2q^m \end{cases} \tag{4.5}$$
or

\[
\begin{align*}
C^N \pm x &= 2^{3m-1} q^m \\
C^N \mp x &= 2. 
\end{align*}
\tag{4.6}
\]

First consider case (4.5). Adding these two equations yields

\[2^{3m-2} + q^m = (2q + 1)^N,
\]
which has no solutions by Lemma 1, (1).

Next consider case (4.6). Adding these two equations yields

\[2^{3m-2} q^m + 1 = (2q + 1)^N,
\]
which has only the solution \((m, N) = (1, 1)\) by Lemma 1, (2). Hence equation (1.1) has only the solution \((x, m, n) = (c - 1,1, 2)\).

(vi) Let \((x, m, n)\) be a solution of equation (1.1). Suppose that our assumptions are all satisfied.

Put \(q = p^k\) with \(p^k \not\equiv 5 \pmod{8}\) and \(C = 4q + 1\). Then taking equation (1.1) modulo 8 implies that \(1 \equiv 5^n \pmod{8}\), so \(n\) is even, say \(n = 2N\). From (1.1), we have

\[(2q)^{4m} = (C^N + x)(C^N - x).
\]

Since \(\gcd(C^N + x, C^N - x) = 2\) and \(q = p^k\), we obtain the following two cases:

or

\[
\begin{align*}
C^N \pm x &= 2^{4m-1} q^m \\
C^N \mp x &= 2
\end{align*}
\tag{4.7}
\]

First consider case (4.7). Adding these two equations yields

\[2^{4m-2} + q^m = (4q + 1)^N,
\]
which has no solutions by Lemma 2, (1).

Next consider case (4.8). Adding these two equations yields

\[2^{4m-2} q^m + 1 = (4q + 1)^N,
\]
which has only the solution \((m, N) = (1, 1)\) by Lemma 2, (2). Hence equation (1.1) has only the solution \((x, m, n) = (c - 1,1, 2)\). This completes the proof of Theorem 1.

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References


[Lj] W. Ljunggren, Some theorems on indeterminate equations of the form \( \frac{x^n - 1}{x - 1} = y^q \) (Norwegian), Norsk Mat. Tidsskr., 25 (1943), 17–20.


[To1] M. Toyoizumi, On the diophantine equation \( y^2 + D^m = 2^n \), Commentarii mathematici Universitatis Sancti Pauli, 27 (1979), 105–111.


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