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Sequences of Generalized Bounded Variation

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Abstract

The purpose of this article is to offer a short, guided tour through the introduction and development of a class of sequence spaces that represent a discretization of spaces of functions of generalized bounded variation. The paths that we follow are motivated by, and parallel, some of those forged over the last 140 years in expanding Jordan's original concept of functions of bounded variation on a compact interval. In addition, we devote attention to the issue of stability of the sequence spaces under coordinatewise multiplication and also endow them with a canonical norm.

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1 Introduction

The point of departure for this article is the collection bv of sequences of *bounded variation*: a (real or) complex sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ satisfies $\mathbf{x} \in bv$ if its *variation* $\sum_{n=1}^{\infty} |x_{n+1} - x_n|$ is finite. It is straightforward to confirm that bv is a linear space with respect to coordinatewise addition and scalar multiplication.

We proceed to offer incremental generalizations of the sequence space bv by retracing the avenues pursued by Weiner [8], Young [10], and Waterman [7] in their work with *functions of bounded variation*. Indeed, since their introduction by Jordan in the late 19-th century, the theory surrounding functions on a compact interval having bounded variation has been richly developed and widely extended, most notably on account of their connection to the convergence of Fourier series. See, for example, [2] and [5] for historical summaries and numerous references.

There are two immediate drawbacks to this discretization of the classical function definitions: the spaces that arise *ob ovo* contain unbounded sequences and they also fail to be stable under coordinatewise multiplication. Much of section 2 is devoted to exposing and detailing these downsides. A simple remedy for the first is to restrict attention to *bounded* sequences having finite generalized variation, and we find that stability under coordinatewise multiplication obtains as a lagniappe in a sense to be made precise in Proposition 1 and subsection 2.4.

Our step-by-step presentation is intended to be deliberate and explicative, and most of it is accessible to readers familiar with little more than the basics of sequences and series from calculus; only in Proposition 3 do we encounter the more advanced notions of closedness and convergence in a normed linear space. Students in search of an activity to enhance their course work in analysis might therefore pause in their reading and supply their own proofs or examples. Moreover, the article could readily be redacted to serve as an outline for a guided discovery project or enrichment exercises. At the very least, we hope to hereby contribute some modest additions to the analyst's arsenal of special spaces, such as those itemized in Chapter IV.2 of [3].

In the third and final section, we consider some of the relationships among the various sequence spaces that we encountered along the way.

2 Generalized variations

2.1 p-variation

Paralleling the definitions given by Wiener [8] and Young [10] for functions on a compact interval, for any real number $p \geq 1$, we define the *p-variation* of a sequence $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ of complex numbers by

$$\mathrm{var}_p(\mathbf{x}) = \left(\sum_{n=1}^{\infty} |x_{n+1} - x_n|^p \right)^{1/p}.$$

The set of all sequences having finite p-variation constitutes a linear space; in particular, if the p-variations of the sequences $\mathbf{x}$ and $\mathbf{y}$ are both finite, then, by *Minkowski's inequality,* $\mathrm{var}_p(\mathbf{x} + \mathbf{y}) \leq \mathrm{var}_p(\mathbf{x}) + \mathrm{var}_p(\mathbf{y})$.

The spaces of finite p-variation corresponding to values of $p > 1$ stand in stark contrast to the space *bv* which arises when $p = 1$. For example, a telescoping argument confirms that sequences of ordinary bounded variation are Cauchy, and hence convergent. In particular, all sequences in *bv* are bounded, whereas, for any $p > 1$, the unbounded sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ given by

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \qquad \text{for all } n \in \mathbb{N}$$

has finite p-variation since the series

$$\operatorname{var}_p(\mathbf{x})^p = \sum_{n=1}^{\infty} |x_{n+1} - x_n|^p = \sum_{n=1}^{\infty} \frac{1}{(n+1)^p}$$

converges. The sequential environment differs in this regard from the traditional setting of functions on a compact interval since functions of bounded p-variation on a compact interval are necessarily bounded (Lemma 1.6 of [9]).

For $p > 1$, there are also bounded, divergent sequences having finite p-variation. For example, for each $n \geq 2$, consider the 2^n-tuple whose entries are the vertices, starting with 1 and proceeding counterclockwise, of the regular 2^n-gon that is inscribed in the unit circle of the complex plane. Assemble the sequence $\mathbf{z}$ by juxtaposing these 2^n-tuples in blocks as follows:

$$\underbrace{1, i, -1, -i}_{\text{4 terms}}, \underbrace{1, e^{\frac{\pi i}{4}}, i, e^{\frac{3\pi i}{4}}, -1, e^{\frac{5\pi i}{4}}, -i, e^{\frac{7\pi i}{4}}}_{\text{8 terms}}, \underbrace{1, e^{\frac{\pi i}{8}}, e^{\frac{2\pi i}{8}}, e^{\frac{3\pi i}{8}}, i, \ldots, -1, \ldots, -i, \ldots, e^{\frac{7\pi i}{8}}}_{\text{16 terms}}, \ldots .$$

Far from being convergent, every number on the unit circle in the complex plane is a partial limit of the sequence $\mathbf{z}$. Moreover, for $p > 1$,

$$\operatorname{var}_p(\mathbf{z})^p = \sum_{n=1}^{\infty} |z_{n+1} - z_n|^p = \sum_{n=2}^{\infty} 2^n \left| \exp\left(\frac{2\pi i}{2^n}\right) - 1 \right|^p < \sum_{n=2}^{\infty} 2^n \left(\frac{2\pi}{2^n}\right)^p = (2\pi)^p \sum_{n=2}^{\infty} \left(2^{1-p}\right)^n,$$

where the inequality above results from comparing the length of a side of the inscribed 2^n-gon to that of the arc of the unit circle that it subtends. Since the last series above converges, $\operatorname{var}_p(\mathbf{z})$ is finite.

The classical sequence space bv is also stable under coordinatewise multiplication, but this additional algebraic structure is unique to the case $p = 1$. To see why, fix $p > 1$ and consider the sequence $\mathbf{x}$ defined by

$$x_n = n^{\frac{p-1}{2p}} \qquad \text{for all } n \in \mathbb{N}.$$

We aim to show that $\operatorname{var}_p(\mathbf{x})$ is finite, but that $\operatorname{var}_p(\mathbf{x}^2)$ is infinite.

The mean value theorem applied to the function f defined by $f(u) = u^{\frac{p-1}{2p}}$ for all $u \geq 1$ ensures the existence of numbers $c_n \in (n, n+1)$ that satisfy $f(n+1) - f(n) = f'(c_n)$ for all $n \in \mathbb{N}$. We thus have the following estimate:

$$\begin{aligned}\operatorname{var}_p(\mathbf{x})^p = \sum_{n=1}^{\infty} \left[(n+1)^{\frac{p-1}{2p}} - n^{\frac{p-1}{2p}}\right]^p = \sum_{n=1}^{\infty} \left[\frac{p-1}{2p}(c_n)^{\frac{-p-1}{2p}}\right]^p = \left(\frac{p-1}{2p}\right)^p \sum_{n=1}^{\infty} \left(\frac{1}{c_n}\right)^{\frac{p+1}{2}} \\ < \left(\frac{p-1}{2p}\right)^p \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\frac{p+1}{2}}.\end{aligned}$$

The fact that $p > 1$ ensures that the exponent $(p+1)/2 > 1$ from which it follows that the last series above converges. Consequently, $\mathrm{var}_p(\mathbf{x})$ is finite.

It remains to show is that the p-variation of the sequence $\mathbf{x}^2$ is not finite. To this end, the auxiliary function g defined by $g(u) = u^{1-\frac{1}{p}}$ for all $u \geq 1$ is helpful. This time, the mean value theorem provides numbers $d_n \in (n, n+1)$ for which $g(n+1) - g(n) = g'(d_n)$ for all $n \in \mathbb{N}$. Thus,

$$\mathrm{var}_p(\mathbf{x}^2)^p = \sum_{n=1}^{\infty} \Big[(n+1)^{\frac{p-1}{p}} - n^{\frac{p-1}{p}}\Big]^p = \sum_{n=1}^{\infty} \Big[\Big(\frac{p-1}{p}\Big)(d_n)^{-1/p}\Big]^p = \Big(\frac{p-1}{p}\Big)^p \sum_{n=1}^{\infty} \frac{1}{d_n}$$
$$> \Big(\frac{p-1}{p}\Big)^p \sum_{n=1}^{\infty} \frac{1}{n+1}.$$

Since the last series above diverges, we conclude that $\mathrm{var}_p(\mathbf{x}^2)$ is infinite.

The following proposition summarizes and formalizes the preceding discussion.

Proposition 1 *If* $\mathbf{x}$ *is a sequence of complex numbers and* $p \geq 1$, *then the following assertions are equivalent:*

(a) $p = 1$;

(b) *if* $\mathrm{var}_p(\mathbf{x}) < \infty$, *then* $\mathbf{x}$ *is convergent;*

(c) *if* $\mathrm{var}_p(\mathbf{x}) < \infty$, *then* $\mathbf{x}$ *is bounded;*

(d) *the space* $\{\mathbf{x} : \mathrm{var}_p(\mathbf{x}) < \infty\}$ *is stable under coordinatewise multiplication.*

2.2 Λ-variation

In the 1972 article [7], Waterman introduced weights to the terms that make up the original definition of the variation of a function on a compact interval. Following his lead, we here let $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ be a sequence of real numbers that is *bounded away from zero* in the sense that there is a $c > 0$ for which $\lambda_n \geq c$ for all $n \in \mathbb{N}$. The Λ-*variation* of a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is then defined by

$$\mathrm{var}_\Lambda(\mathbf{x}) = \sum_{n=1}^{\infty} \frac{|x_{n+1} - x_n|}{\lambda_n}.$$

Here again, it is easy to check that the set of all sequences having finite Λ-variation constitutes a linear space. The space *bv* clearly results from the choice $\Lambda = \mathbf{1}$, the constant sequence whose terms are all ones, or, indeed, whenever Λ is any constant sequence. In fact, as we will see in section 3.1, in this context, any bounded sequence Λ gives rise to *bv*.

If the sequence Λ is unbounded, then, as is the case for the p-variation when $p > 1$, there are unbounded sequences having finite Λ-variation. For example, with the choice $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ defined by $\lambda_n = n$ for all $n \in \mathbb{N}$, the unbounded sequence $\mathbf{x} = (1, \sqrt{2}, \sqrt{3}, ...)$ has finite Λ-variation since

$$\mathrm{var}_\Lambda(\mathbf{x}) = \sum_{n=1}^{\infty} \frac{|x_{n+1} - x_n|}{\lambda_n} = \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n(\sqrt{n+1} + \sqrt{n})} < \infty.$$

Computation of the Λ-variation of the sequence $\mathbf{x}^2 = (1, 2, 3, ...)$, however, yields:

$$\mathrm{var}_\Lambda(\mathbf{x}^2) = \sum_{n=1}^{\infty} \frac{|x_{n+1}^2 - x_n^2|}{\lambda_n} = \sum_{n=1}^{\infty} \frac{(n+1) - n}{n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

a divergent series. Since the Λ-variation of the sequence $\mathbf{x}$ is finite but the Λ-variation of its square $\mathbf{x}^2$ is not, the space of sequences of finite Λ-variation is not stable under coordinatewise multiplication. This, again, is in contrast to the setting of functions on an interval, in which case Waterman's *functions of harmonic bounded variation* are automatically bounded [7].

We shall explore further the impact of various choices for Λ in subsection 3.1.

2.3 Λ^p-variation

Our ultimate generalization of the classical space *bv* of sequences of bounded variation combines the ingredients of the previous two subsections as follows: for $p \geq 1$ and a sequence Λ of positive numbers that is bounded away from zero, the Λ^p-*variation* of the sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is defined by

$$\mathrm{var}_{\Lambda^p}(\mathbf{x}) = \left(\sum_{n=1}^{\infty} \frac{|x_{n+1} - x_n|^p}{\lambda_n} \right)^{1/p}.$$

The spaces of sequences whose Λ^p-variations are finite are discrete analogues of the Waterman-Shiba spaces that were introduced in 1980 by Shiba [6]. Not surprisingly, these sequence spaces inherit the shortcomings of the spaces of sequences having finite p- and Λ-variations from which they are born, namely, the lurking presence of unbounded sequences and instability under coordinatewise multiplication.

2.4 The algebras Λ^p-*bv*

To clear these obstacles, we hereafter narrow our focus to *bounded* sequences having finite Λ^p-variations. The resulting space will be denoted Λ^p-*bv*; thus, for $p \geq 1$ and a sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ of positive numbers that is bounded away from zero, the sequence $\mathbf{x} \in \Lambda^p\text{-}bv$ if and only if $\mathbf{x} \in \ell^\infty$ and $\mathrm{var}_{\Lambda^p}(\mathbf{x}) < \infty$. This restriction allows for the definition

$$\|\mathbf{x}\|_{\Lambda^p} = \|\mathbf{x}\|_\infty + \mathrm{var}_{\Lambda^p}(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \Lambda^p\text{-}bv,$$

which renders the pair $(\Lambda^p\text{-}bv, \|\cdot\|_{\Lambda^p})$ a normed linear space. It turns out that if $\mathbf{x}$ and $\mathbf{y}$ are sequences in $\Lambda^p\text{-}bv$, then their product $\mathbf{xy} \in \Lambda^p\text{-}bv$ and $\|\mathbf{xy}\|_{\Lambda^p} \le \|\mathbf{x}\|_{\Lambda^p}\|\mathbf{y}\|_{\Lambda^p}$. Moreover, the norm $\|\cdot\|_{\Lambda^p}$ is complete, so that the pair $(\Lambda^p\text{-}bv, \|\cdot\|_{\Lambda^p})$ is a commutative Banach algebra whose multiplicative identity element is the constant sequence $\mathbf{1}$. Additionally, $\|\mathbf{1}\|_{\Lambda^p} = 1$. All this may be confirmed by first principles, or, alternatively, as detailed in Example 1 of [4].

3 Relationships among the sequence algebras

This section features some relationships involving the algebras $\Lambda^p\text{-}bv$. Since $\mathbf{1}$ represents the sequence consisting entirely of ones, the notation $\mathbf{1}^p\text{-}bv$ denotes the algebra of bounded sequences $\mathbf{x} = (x_1, x_2, \dots)$ for which the series $\sum_{n=1}^\infty |x_{n+1} - x_n|^p$ converges. We begin with the following rudimentary inclusions.

Proposition 2 *If $1 \le p < q$ and the sequence Λ is bounded away from zero, then the following inclusions hold:*

$$\mathbf{1}^p\text{-}bv \subseteq \Lambda^p\text{-}bv \subseteq \Lambda^q\text{-}bv. \qquad (\star)$$

Proof. Since the sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ is bounded away from zero, there is a number $c > 0$ for which $c \le \lambda_n$ for all $n \in \mathbb{N}$. Now, let $\mathbf{x} \in \mathbf{1}^p\text{-}bv$. Then

$$\mathrm{var}_{\Lambda^p}(\mathbf{x})^p = \sum_{n=1}^\infty \frac{|x_{n+1} - x_n|^p}{\lambda_n} \le \sum_{n=1}^\infty \frac{|x_{n+1} - x_n|^p}{c} = \frac{1}{c}\,\mathrm{var}_{\mathbf{1}^p}(\mathbf{x})^p < \infty.$$

Thus, $\mathbf{x} \in \Lambda^p\text{-}bv$, which establishes the first inclusion.

Next, if $\mathbf{x} \in \Lambda^p\text{-}bv$, then computation of $\mathrm{var}_{\Lambda^q}(\mathbf{x})^q$ yields

$$\sum_{n=1}^\infty \frac{|x_{n+1} - x_n|^q}{\lambda_n} = \sum_{n=1}^\infty \frac{|x_{n+1} - x_n|^{q-p}\,|x_{n+1} - x_n|^p}{\lambda_n} \le (2\|\mathbf{x}\|_\infty)^{q-p} \sum_{n=1}^\infty \frac{|x_{n+1} - x_n|^p}{\lambda_n} < \infty,$$

which implies that $\mathbf{x} \in \Lambda^q\text{-}bv$ to verify the second inclusion. □

As a consequence of Proposition 2, the algebra $bv = \mathbf{1}^1\text{-}bv$ is contained in all of the algebras $\Lambda^p\text{-}bv$ of generalized bounded variation under consideration. The algebras bv and ℓ^∞ are thus extremal in the sense that $bv \subseteq \Lambda^p\text{-}bv \subseteq \ell^\infty$ for all admissible Λ and all $p \ge 1$. The possibilities for the equalities $bv = \Lambda^p\text{-}bv$ or $\Lambda^p\text{-}bv = \ell^\infty$ are addressed below in subsection 3.1. But first we incorporate the familiar ℓ^p algebras into the picture.

Proposition 3 *For $p \geq 1$ and a sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ of positive numbers that is bounded away from zero, the algebra ℓ^p is a proper, non-closed ideal of the Banach algebra $(\Lambda^p\text{-}bv, \|\cdot\|_{\Lambda^p})$.*

Proof. If $\mathbf{x} = (x_1, x_2, x_3, \dots) \in \ell^p$, then $\tilde{\mathbf{x}} = (x_2, x_3, x_4, \dots) \in \ell^p$. Thus, the difference $\tilde{\mathbf{x}} - \mathbf{x} = (x_2 - x_1, x_3 - x_2, x_3 - x_4, \dots) \in \ell^p$ which implies that $\mathbf{x} \in \mathbf{1}^p\text{-}bv$. Consequently, $\ell^p \subseteq \mathbf{1}^p\text{-}bv$. The inclusion is, in fact, proper since, for example, the multiplicative identity element $\mathbf{1} \in \mathbf{1}^p\text{-}bv \setminus \ell^p$. By Proposition 2, we deduce that $\ell^p \subsetneq \Lambda^p\text{-}bv$.

To confirm that ℓ^p is an ideal of $\Lambda^p\text{-}bv$, let $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \ell^p$ and $\mathbf{y} = (y_n)_{n\in\mathbb{N}} \in \Lambda^p\text{-}bv$. By definition, $\Lambda^p\text{-}bv \subseteq \ell^\infty$ so $\sum_{n=1}^\infty |x_n y_n|^p \leq \|y\|_\infty^p \sum_{n=1}^\infty |x_n|^p < \infty$ which implies that the product $\mathbf{xy} \in \ell^p$.

To establish that ℓ^p is not $\|\cdot\|_{\Lambda^p}$-closed in $\Lambda^p\text{-}bv$, for $n \in \mathbb{N}$, let $\mathbf{x}_n$ denote the sequence whose terms are

$$\underbrace{1,\ 1/2^{1/p},\ 1/3^{1/p},\ \dots\ 1/n^{1/p}}_{n \text{ terms}},\ 0,\ 0,\ 0\dots$$

and let $\mathbf{x} = (1/n^{1/p})_{n\in\mathbb{N}}$. Then $\mathbf{x}_n \in \ell^p$ for each $n \in \mathbb{N}$, but $\mathbf{x} \notin \ell^p$. To prove that $\mathbf{x} \in \Lambda^p\text{-}bv$, we first treat the case $p = 1$. Because the sequence Λ is bounded away from zero, there is a number c that satisfies $0 < c \leq \lambda_n$ for all $n \in \mathbb{N}$. We thus have that

$$\mathrm{var}_{\Lambda^1}(\mathbf{x}) = \sum_{n=1}^\infty \frac{1}{\lambda_n}\left|\frac{1}{n+1} - \frac{1}{n}\right| = \sum_{n=1}^\infty \frac{1}{\lambda_n n(n+1)} < \frac{1}{c}\sum_{n=1}^\infty \frac{1}{n(n+1)} < \infty,$$

which implies that $\mathbf{x} \in \Lambda^p\text{-}bv$ when $p = 1$. Next, for $p > 1$, the mean value theorem applied to the function f defined by $f(u) = u^{1/p}$ for all $u \geq 1$ supplies, for each $n \in \mathbb{N}$, a number $d_n \in (n, n+1)$ for which $f(n+1) - f(n) = f'(d_n)$. It follows that

$$\begin{aligned}\mathrm{var}_{\Lambda^p}(\mathbf{x})^p &= \sum_{n=1}^\infty \frac{1}{\lambda_n}\left|\frac{1}{(n+1)^{1/p}} - \frac{1}{n^{1/p}}\right|^p = \sum_{n=1}^\infty \frac{\left[(n+1)^{1/p} - n^{1/p}\right]^p}{\lambda_n n(n+1)} = \frac{1}{p^p}\sum_{n=1}^\infty \frac{d_n^{1-p}}{\lambda_n n(n+1)}\\ &< \frac{1}{cp^p}\sum_{n=1}^\infty \frac{n^{1-p}}{n(n+1)} = \frac{1}{cp^p}\sum_{n=1}^\infty \frac{1}{n^p(n+1)} < \infty,\end{aligned}$$

so that $\mathbf{x} \in \Lambda^p\text{-}bv$. Finally, to establish the $\|\cdot\|_{\Lambda^p\text{-}bv}$-convergence of the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ to $\mathbf{x}$, observe that

$$\begin{aligned}\|\mathbf{x} - \mathbf{x}_n\|_{\Lambda^p} &= \|\mathbf{x} - \mathbf{x}_n\|_\infty + \mathrm{var}_{\Lambda^p}(\mathbf{x} - \mathbf{x}_n)\\ &= \frac{1}{(n+1)^{1/p}} + \frac{1}{\lambda_n (n+1)^{1/p}} + \left(\sum_{k=n+1}^\infty \frac{1}{\lambda_k}\left|\frac{1}{(k+1)^{1/p}} - \frac{1}{k^{1/p}}\right|^p\right)^{1/p}.\end{aligned}$$

The first two terms above clearly tend to zero as n tends to infinity, and the fact that $\mathbf{x} \in \Lambda^p\text{-}bv$ implies that the summation term does also. Consequently,

$\|\mathbf{x}-\mathbf{x}_n\|_{\Lambda^p} \to 0$ as $n \to \infty$. The $\|\cdot\|_{\Lambda^p}$-limit $\mathbf{x}$ of the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ of elements of ℓ^p thus lies in Λ^p-$bv \setminus \ell^p$, so that the ideal ℓ^p is not $\|\cdot\|_{\Lambda^p}$-closed in Λ^p-bv. $\square$

3.1 The sequence Λ

In this subsection, we refine the inclusions $bv \subseteq \Lambda^p$-$bv \subseteq \ell^\infty$ based on the nature of the sequence Λ.

Proposition 4 *For $p \geq 1$ and a sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ of positive numbers that is bounded away from zero, the following assertions hold:*

(a) *$\mathbf{1}^p$-$bv = \Lambda^p$-bv if and only if the sequence Λ is bounded;*

(b) *Λ^p-$bv = \ell^\infty$ if and only if the series $\sum_{n=1}^\infty 1/\lambda_n$ is convergent.*

In particular, the equality $bv = \Lambda^p$-bv holds if and only if $p = 1$ and Λ is bounded.

Proof. (a) By Proposition 2, we already have that $\mathbf{1}^p$-$bv \subseteq \Lambda^p$-bv. So suppose that the number $M > 0$ satisfies $\lambda_n \leq M$ for all $n \in \mathbb{N}$, and let $\mathbf{x} \in \Lambda^p$-bv. Then

$$\mathrm{var}_{\mathbf{1}^p}(\mathbf{x})^p = \sum_{n=1}^\infty |x_{n+1}-x_n|^p \leq M \sum_{n=1}^\infty \frac{|x_{n+1}-x_n|^p}{\lambda_n} = M\mathrm{var}_{\Lambda^p}(\mathbf{x})^p < \infty,$$

which implies that $\mathbf{x} \in \mathbf{1}^p$-$bv$. Thus, Λ^p-$bv \subseteq \mathbf{1}^p$-$bv$.

Conversely, if the sequence Λ is unbounded, then we may and do extract a non-decreasing subsequence $(\lambda_{n_k})_{k\in\mathbb{N}}$ for which $\lambda_{n_k} > k^p 2^k$ and also $n_{k+1}-n_k > 1$ for all $k \in \mathbb{N}$. Define the sequence $\mathbf{x}$ by $x_{n_k+1} = k^p$ for all $k \in \mathbb{N}$, and otherwise the terms of $\mathbf{x}$ are 0. Computation of the p-th power of the Λ^p-variation of $\mathbf{x}$ yields

$$\begin{aligned}
\mathrm{var}_{\Lambda^p}(\mathbf{x})^p = \sum_{n=1}^\infty \frac{|x_{n+1}-x_n|^p}{\lambda_n} &= \frac{1^p}{\lambda_{n_1}} + \frac{1^p}{\lambda_{n_1+1}} + \frac{2^p}{\lambda_{n_2}} + \frac{2^p}{\lambda_{n_2+1}} + \frac{3^p}{\lambda_{n_3}} + \frac{3^p}{\lambda_{n_3+1}} + \cdots \\
&\leq \frac{1^p}{\lambda_{n_1}} + \frac{1^p}{\lambda_{n_1}} + \frac{2^p}{\lambda_{n_2}} + \frac{2^p}{\lambda_{n_2}} + \frac{3^p}{\lambda_{n_3}} + \frac{3^p}{\lambda_{n_3}} + \cdots \\
&= 2\sum_{k=1}^\infty \frac{k^p}{\lambda_{n_k}} < 2\sum_{k=1}^\infty \frac{1}{2^k} = 2,
\end{aligned}$$

where the inequality results from the subsequence $(\lambda_{n_k})_{k\in\mathbb{N}}$ being non-decreasing. It follows that $\mathbf{x} \in \Lambda^p$-bv. But a glance at the first line of the preceding display reveals that if $\Lambda = \mathbf{1}$, then

$$\mathrm{var}_{\mathbf{1}^p}(\mathbf{x})^p = \sum_{n=1}^\infty |x_{n+1}-x_n|^p = 1^p + 1^p + 2^p + 2^p + 3^p + 3^p + \cdots,$$

so that $\mathbf{x} \notin \mathbf{1}^p\text{-}bv$.

(b) First, suppose that the equality $\Lambda^p\text{-}bv = \ell^\infty$ holds. Then, in particular, the bounded sequence $\mathbf{x}$ whose terms are defined by $x_n = 1$ if n is odd and $x_n = 0$ if n is even satisfies $\mathbf{x} \in \Lambda^p\text{-}bv$. But

$$\operatorname{var}_{\Lambda^p}(\mathbf{x})^p = \sum_{n=1}^{\infty} \frac{|x_{n+1} - x_n|^p}{\lambda_n} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n},$$

so the series $\sum_{n=1}^{\infty} 1/\lambda_n$ is convergent.

Conversely, suppose that the series $\sum_{n=1}^{\infty} 1/\lambda_n$ is convergent and let $\mathbf{x} \in \ell^\infty$. Then

$$\operatorname{var}_{\Lambda^p}(\mathbf{x})^p = \sum_{k=1}^{\infty} \frac{|x_{k+1} - x_k|^p}{\lambda_k} \le \sum_{k=1}^{\infty} \frac{(|x_{k+1}| + |x_k|)^p}{\lambda_k} \le 2^p \|\mathbf{x}\|_\infty^p \sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty,$$

which implies that $\mathbf{x} \in \Lambda^p\text{-}bv$. Since $\Lambda^p\text{-}bv \subseteq \ell^\infty$ by definition, the equality $\Lambda^p\text{-}bv = \ell^\infty$ holds.

Turning to the final assertion, if $p = 1$ and Λ is bounded, then, by part (a), it follows that $bv = \mathbf{1}^p\text{-}bv = \Lambda^p\text{-}bv$.

For the converse, if Λ is unbounded, then part (a), together with the first inclusion in $(\star)$, imply that the proper inclusion $\mathbf{1}^p\text{-}bv \subsetneq \Lambda^p\text{-}bv$ holds for all $p \ge 1$. In particular, $bv = \mathbf{1}^1\text{-}bv \subsetneq \Lambda^1\text{-}bv$. The second inclusion in $(\star)$ then ensures the proper containment $bv \subsetneq \Lambda^p\text{-}bv$ for all $p \ge 1$.

Lastly, for $p > 1$, the sequence $\mathbf{z}$ from subsection 2.1 satisfies $\mathbf{z} \in \mathbf{1}^p\text{-}bv \setminus bv$ and hence, again by the first inclusion in $(\star)$, it follows that $\mathbf{z} \in \Lambda^p\text{-}bv \setminus bv$. □

A propos of Proposition 4, we remark that if $\mathbf{1}^p\text{-}bv = \Lambda^p\text{-}bv$, then, by *Johnson's uniqueness-of-norm theorem* (Corollary 5.29 of [1]), the Banach algebra norms $\|\cdot\|_{\mathbf{1}^p}$ and $\|\cdot\|_{\Lambda^p}$ on $\Lambda^p\text{-}bv$ are equivalent, and, similarly, if $\Lambda^p\text{-}bv = \ell^\infty$, then the norms $\|\cdot\|_{\Lambda^p}$ and $\|\cdot\|_\infty$ on $\Lambda^p\text{-}bv$ are equivalent.

3.2 A couple more relationships

Section 2 of [5] is devoted to detailing several interesting interleaving relationships between various spaces of generalized variation in the context of real-valued functions on a compact interval. Discrete analogues of these hold in the sequence setting as well, and the proofs largely proceed *mutatis mutandis*. We here simply highlight two of these results in which our arguments are slightly different.

In the case that $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ is the sequence defined by $\lambda_n = n$ for all $n \in \mathbb{N}$, the resulting algebra $hbv = \Lambda^1\text{-}bv$ is the algebra of *sequences of harmonic bounded variation*. Thus, a sequence $\mathbf{x} = (x_1, x_2, \dots) \in hbv$ precisely when $\mathbf{x} \in \ell^\infty$ and the series $\sum_{n=1}^{\infty} |x_{n+1} - x_n|/n$ converges. The algebra hbv is evidently an appropriate discretization of Waterman's class discussed in subsection 2.2. By Proposition 4, the proper inclusions $bv \subsetneq hbv \subsetneq \ell^\infty$ hold. In fact, we have the following result – an analogue of the penultimate display of [5].

Proposition 5 $\bigcup_{p\geq 1} \mathbf{1}^p\text{-}bv \subsetneq hbv$

Proof. Since $bv \subsetneq hbv$, let $p > 1$, and let q be the conjugate exponent of p. So $q > 1$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that the sequence $\mathbf{x} = (x_1, x_2, \dots) \in \mathbf{1}^p\text{-}bv$; thus $\mathbf{x} \in \ell^\infty$ and the series $\sum_{n=1}^\infty |x_{n+1} - x_n|^p$ converges. By Hölder's inequality,

$$\sum_{n=1}^{\infty} \frac{|x_{n+1} - x_n|}{n} \leq \left(\sum_{n=1}^{\infty} |x_{n+1} - x_n|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} \frac{1}{n^q} \right)^{1/q},$$

and since both series on the right hand side converge, it follows that $\mathbf{x} \in hbv$.

To verify that the inclusion is proper, introduce the sequence $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ defined by $a_n = 1$ if n is a perfect square and $a_n = 0$ otherwise. Then $\mathbf{a} \in hbv$ because $\mathbf{a} \in \ell^\infty$ and the series

$$\sum_{n=1}^{\infty} \frac{|a_{n+1} - a_n|}{n} = \frac{1}{1} + \frac{0}{2} + \frac{1}{3} + \frac{1}{4} + \frac{0}{5} + \frac{0}{6} + \frac{0}{7} + \frac{1}{8} + \frac{1}{9} + \frac{0}{10} + \cdots = 1 + \sum_{n=2}^{\infty} \left(\frac{1}{n^2 - 1} + \frac{1}{n^2} \right)$$

converges. But for each $p \geq 1$, the sequence $\mathbf{a} \notin \mathbf{1}^p\text{-}bv$ because the series

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n|^p = 1 + 0 + 1 + 1 + 0 + 0 + 0 + 1 + 1 + 0 + \cdots$$

diverges. □

Finally, the following is an analogue of Theorem 6 of [5].

Proposition 6 *Suppose that* $p \geq 1$ *and define the sequence* $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ *by* $\lambda_n = n^{1-\frac{1}{p}}$ *for all* $n \in \mathbb{N}$. *Then* $\Lambda^1\text{-}bv \subsetneq \mathbf{1}^p\text{-}bv$.

Proof. Suppose that $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \Lambda^1\text{-}bv$, and, for $n \in \mathbb{N}$, let $a_n = |x_{n+1} - x_n|$ and $b_n = n^{1/p} a_n$. Then

$$a_n^p = \frac{b_n}{n} \cdot b_n^{p-1}.$$

By hypothesis, the series $\sum_{n=1}^\infty b_n/n$ converges which implies that the sequence $(b_n)_{n\in\mathbb{N}}$ is bounded. Thus, the series

$$\sum_{n=1}^{\infty} \frac{b_n}{n} \cdot b_n^{p-1} = \sum_{n=1}^{\infty} a_n^p = \sum_{n=1}^{\infty} |x_{n+1} - x_n|^p$$

converges. Consequently, the sequence $\mathbf{x} \in \mathbf{1}^p\text{-}bv$ to complete the proof. □

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