Hypercyclicity of the Adjoint of Weighted Composition Operators on the Reproducing Kernel Hilbert Spaces

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Abstract

The aim of this paper is to study the hypercyclicity of the adjoint of weighted composition operators on the vector-valued analytic reproducing kernel Hilbert spaces.

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1 Introduction

Denote by $\mathbb{D}$ the open unit disk in the complex plane $\mathbb{C}$. Let $\mathcal{E}$ be a Hilbert space and $\mathcal{L}(\mathcal{E})$ the set of bounded linear operators on $\mathcal{E}$. An operator-valued function $K : \mathbb{D} \times \mathbb{D} \to \mathcal{L}(\mathcal{E})$ is called an analytic kernel (cf. [6]) if for any fixed $w \in \mathbb{D}$, the operator-valued function $K(\cdot, w) : \mathbb{D} \to \mathcal{L}(\mathcal{E})$ is analytic, and

$$\sum_{i,j=1}^{n} \langle K(w_i, w_j)\eta_j, \eta_i \rangle_{\mathcal{E}} \geq 0,$$

for all $\{w_i\}_{i=1}^{n} \subset \mathbb{D}$, $\{\eta_i\}_{i=1}^{n} \subset \mathcal{E}$ and $n \in \mathbb{N}^+$. In this case, by the Moore’s Theorem ([7]), there exists a Hilbert space $\mathcal{H}_{\mathcal{E}}(K)$ of $\mathcal{E}$-valued analytic functions on $\mathbb{D}$ such that $\{K(\cdot, w)\eta : w \in \mathbb{D}, \eta \in \mathcal{E}\}$ is a total set in $\mathcal{H}_{\mathcal{E}}(K)$, and we call $\mathcal{H}_{\mathcal{E}}(K)$ the vector-valued analytic reproducing kernel Hilbert spaces.
Let $\psi$ be a multiplier of $\mathcal{H}_E(K)$, i.e., $\psi$ is a complex-valued function on $\mathbb{D}$ satisfying $\psi \cdot \mathcal{H}_E(K) \subset \mathcal{H}_E(K)$. Suppose that $\varphi$ is an analytic self-map of $\mathbb{D}$ such that $(f \circ \varphi)(z) = f(\varphi(z)) \in \mathcal{H}_E(K)$, then the weighted composition operator $C_{\varphi,\psi}$ is defined by

$$(C_{\varphi,\psi}f)(z) = \psi(z)f(\varphi(z)),$$

for $f \in \mathcal{H}_E(K)$ and $z \in \mathbb{D}$. The closed graph theorem shows that $C_{\varphi,\psi} : \mathcal{H}_E(K) \rightarrow \mathcal{H}_E(K)$ is bounded, so the adjoint of weighted composition operator $C_{\varphi,\psi}^*$ is also bounded on $\mathcal{H}_E(K)$.

For a Banach space $X$, an operator $T \in \mathcal{L}(X)$ is said to be hypercyclic if there exists a vector $x \in X$ such that the orbit of $x$ under $T$, $\text{Orb}(x,T) := \{x, Tx, T^2x, \ldots\}$ is dense in $X$, and $x$ is called the hypercyclic vector for $T$. In [2], Bourdon and Shapiro studied thoroughly the hypercyclicity of the composition operator (see also [3]).

Recently, Mundayadan and Sarkar characterized completely the hypercyclicity, as well other dynamic properties of the adjoint of the multiplication operator by the coordinate function on $\mathcal{H}_E(K)$ in [6]. In 2011, Kamali et al. [5] studied the hypercyclicity of $C_{\varphi,\psi}^*$ acting on the scalar-valued reproducing kernel Hilbert space. In this paper, we concentrate on the more general case, namely the vector-valued analytic reproducing kernel Hilbert spaces $\mathcal{H}_E(K)$.

## 2 Main Results

In this section, we use the following Hypercyclicity Criterion to give the sufficient conditions for $C_{\varphi,\psi}^* : \mathcal{H}_E(K) \rightarrow \mathcal{H}_E(K)$ to be hypercyclic.

**Theorem 2.1.** [1] Let $X$ be a Banach space and $T \in \mathcal{L}(X)$. Suppose that there are dense subsets $\mathcal{D}_1, \mathcal{D}_2 \subset X$, an increasing sequence $(n_k)_{k \in \mathbb{N}}$ of positive integers, and maps $S_{n_k} : \mathcal{D}_2 \rightarrow X$ such that for any $x \in \mathcal{D}_1$, $y \in \mathcal{D}_2$,

(i) $T^{n_k}x \rightarrow 0$, as $k \rightarrow \infty$,

(ii) $S_{n_k}y \rightarrow 0$, as $k \rightarrow \infty$,

(iii) $T^{n_k}S_{n_k}y \rightarrow y$, as $k \rightarrow \infty$.

Then $T$ is hypercyclic.

By [8, Proposition 3.2], we know that if $C_{\varphi,\psi}^*$ is hypercyclic on $\mathcal{H}_E(K)$, then $\varphi$ must be an automorphism. Recall that a sequence $\{c_i\}_{i \in \mathbb{N}}$ of complex numbers is not a Blaschke sequence, if there exists $i_0 \in \mathbb{N}$ such that $|c_i| < 1$ for $i \geq i_0$ and $\sum_{i=1}^{\infty}(1-|c_i|) = \infty$. Moreover, for every $z \in \mathbb{D}$, the linear evaluation
map \( E_z : \mathcal{H}_E(K) \to \mathcal{E} \) given by \( E_z f = f(z) \) is bounded and so \( E_z \circ E_w^* : \mathcal{E} \to \mathcal{E} \) is a bounded linear map. It is easy to verify that \( K(z, w) = E_z \circ E_w^* \) and

\[
\langle f, K(\cdot, w)\eta \rangle_{\mathcal{H}_E(K)} = \langle E_w f, \eta \rangle_{\mathcal{E}}. \tag{1}
\]

**Theorem 2.2.** Let \( \varphi \) be a disk automorphism such that the sets

\[
P = \{ w \in \mathbb{D} : \{ \psi(\varphi_n(w)) \}_{n=0}^{\infty} \text{ is not a Blaschke sequence} \}
\]

and

\[
Q = \{ v \in \mathbb{D} : \{ (\psi(\varphi_{-n}(v)))^{-1} \}_{n=1}^{\infty} \text{ is not a Blaschke sequence} \}
\]

have limit points in \( \mathbb{D} \). If for each \( w \in P \) and each \( v \in Q \) the sequence \( \{ K(\cdot, \varphi_n(w))\eta \}_{n \geq 1} \) and \( \{ K(\cdot, \varphi_{-n}(v))\eta \}_{n \geq 1} \) are bounded for all \( \eta \in \mathcal{E}_0 \), where \( \mathcal{E}_0 \) is a dense subset of \( \mathcal{E} \). Then \( C_{\varphi, \psi}^* \) is hypercyclic on \( \mathcal{H}_E(K) \).

**Proof.** Claim: The sets

\[
\mathcal{M}_P = \text{span}\{ K(\cdot, w)\eta : w \in P, \eta \in \mathcal{E}_0 \}
\]

and

\[
\mathcal{M}_Q = \text{span}\{ K(\cdot, v)\eta : v \in Q, \eta \in \mathcal{E}_0 \}
\]

are dense in \( \mathcal{H}_E(K) \).

Indeed, suppose that \( f \in \mathcal{H}_E(K) \) and \( f \perp K(\cdot, w)\eta \) for all \( w \in P \) and \( \eta \in \mathcal{E}_0 \). Then

\[
0 = \langle f, K(\cdot, w)\eta \rangle_{\mathcal{H}_E(K)} = \langle f(w), \eta \rangle_{\mathcal{E}}.
\]

We have \( f(w) = 0 \) as \( \mathcal{E}_0 \) is dense in \( \mathcal{E} \). Since \( P \) have a limit point in \( \mathbb{D} \), from the identity theorem it follows that \( f \equiv 0 \). Hence \( \mathcal{M}_P \) is dense in \( \mathcal{H}_E(K) \).

Similarly, \( \mathcal{M}_Q \) is also dense in \( \mathcal{H}_E(K) \).

For any \( f \in \mathcal{H}_E(K) \), by (1),

\[
\langle C_{\varphi, \psi}^*(K(\cdot, w)\eta), f \rangle_{\mathcal{H}_E(K)} = \langle K(\cdot, w)\eta, \psi \cdot (f \circ \varphi) \rangle_{\mathcal{H}_E(K)} = \langle \psi(w)f(\varphi(w)), \eta \rangle_{\mathcal{E}} = \langle \overline{\psi(w)}K(\cdot, \varphi(w))\eta, f \rangle_{\mathcal{H}_E(K)}.
\]

Thus we obtain

\[
C_{\varphi, \psi}^*(K(\cdot, w)\eta) = \overline{\psi(w)}K(\cdot, \varphi(w))\eta. \tag{2}
\]
Inductively,
\[ C^{*n}_{\varphi, \psi}(K(\cdot, w)\eta) = \prod_{j=0}^{n-1} \psi(\varphi_j(w)) K(\cdot, \varphi_n(w)) \eta. \]

For \( w \in P \), \( \{ \psi(\varphi_j(w)) \} \) is not a Blaschke sequence and we have
\[ \sum_{j=1}^{\infty} (1 - |\psi(\varphi_j(w))|) = \infty, \]
which is equivalent to
\[ \lim_{n \to \infty} \prod_{j=0}^{n-1} \psi(\varphi_j(w)) = 0. \]

On the other hand, \( \{ K(\cdot, \varphi_n(w)) \eta \}_{n \geq 1} \) is bounded, then we have
\[ \lim_{n \to \infty} C^{*n}_{\varphi, \psi}(K(\cdot, w)\eta) = 0. \]
Therefore \( \lim_{n \to \infty} C^{*n}_{\varphi, \psi} f = 0 \) for any \( f \in \mathcal{M}_P \).

Now suppose that \( \{ K(\cdot, v)\eta : v \in Q, \eta \in \mathcal{E}_0 \} \) is linearly independent. In this case, we can define a linear map
\[ S : \mathcal{M}_Q \to \mathcal{M}_Q \]
by extending the definition
\[ S(K(\cdot, v)\eta) = \psi(\varphi^{-1}(v))^{-1} K(\cdot, \varphi^{-1}(v)) \eta \]
linearly to \( \mathcal{M}_Q \). Since \( \varphi^{-1}(v) \in Q \) whenever \( v \in Q \), \( S(K(\cdot, v)\eta) \in \mathcal{M}_Q \), and we can define \( S^n \) for all \( n \geq 1 \). It is easy to see that
\[ S^n(K(\cdot, v)\eta) = \prod_{j=1}^{n} \psi(\varphi^{-j}(v))^{-1} K(\cdot, \varphi^{-n}(v)) \eta. \]

Since \( \{ (\psi(\varphi^{-j}(v)))^{-1} \} \) is not a Blaschke sequence, we can obtain
\[ \lim_{n \to \infty} \prod_{j=1}^{n} \psi(\varphi^{-j}(v))^{-1} = 0. \]

For \( v \in Q \), \( \{ K(\cdot, \varphi^{-n}(v))\eta \}_{n \geq 1} \) is bounded. It follows that
\[ \lim_{n \to \infty} S^n(K(\cdot, v)\eta) = 0. \]
Hence $\lim_{n \to \infty} S^n f = 0$ for any $f \in M_Q$.
Moreover, if $v \in Q$, then
$$C_{\varphi, \psi}^* S(K(\cdot, v)\eta) = C_{\varphi, \psi}^* \frac{1}{\psi(\varphi^{-1}(v)) \psi(\varphi^{-1}(v))} K(\cdot, \varphi(\varphi^{-1}(v))) \eta$$
$$= K(\cdot, v) \eta,$$
i.e., $C_{\varphi, \psi}^* S = I$ on $M_Q$. We can conclude that $C_{\varphi, \psi}^*$ is hypercyclic on $H_E(K)$ by the Hypercyclicity Criterion.

In the case $\{K(\cdot, v)\eta : v \in Q, \eta \in E_0\}$ is linearly dependent. We adopt the same method in [4, Theorem 4.5] due to Godefroy and Shapiro. Enumerate a countable dense subset $Q_1 = \{v_n : n \geq 1\}$ of $Q$, and inductively choose a subsequence $\{g_n\}_n$ as follows. Take $g_1 = v_1$,
$$Q_2 = Q_1 \setminus \{v \in Q_1 : K(\cdot, v)\eta \in \text{span}\{K(\cdot, g_1)\eta\}\}.$$Denote the first element of $Q_2$ by $g_2$ and let
$$Q_3 = Q_2 \setminus \{v \in Q_2 : K(\cdot, v)\eta \in \text{span}\{K(\cdot, g_1)\eta, K(\cdot, g_2)\eta\}\}.$$Let $g_3$ be the first element of $Q_3$ and continue this process we get a subset $R = \{g_n : n \geq 1\}$ of $Q$ such that the set
$$\{K(\cdot, g)\eta : g \in R, \eta \in E_0\}$$
is linearly independent and
$$M_G = \text{span}\{K(\cdot, g)\eta : g \in R, \eta \in E_0\} = \text{span}\{K(\cdot, v)\eta : v \in Q_1, \eta \in E_0\}$$is dense in $H_E(K)$. Define $S : M_G \to M_G$ as above. Similarly, we also have $C_{\varphi, \psi}^* S = I$ on $M_G$ and $S^n \to 0$ pointwise on $M_G$. Therefore $C_{\varphi, \psi}^*$ is hypercyclic on $H_E(K)$ and the proof is completed.

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References


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