International Mathematical Forum, Vol. 16, 2021, no. 3, 125 – 129
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2021.912248

Strongly Nil Regular Clean Rings

Zubaida M. Ibraheem [*] **and Raghad I. Zidan**

Department of Mathematics
College of Computer Science and Mathematics,
University of Mosul, Mosul, Iraq

Abstract

For an associative ring Ꞧ with identity. We define an element in Ꞧ to be strongly nil regular clean or(strongly ƝꞦ-clean), if it is the sum of a (Von Neumann) regular and a nilpotent (that commute with each other). Ꞧ is called strongly nil regular clean ring if every element of Ꞧ is strongly nil regular clean. In this paper, we introduce some characterization and basic properties of this ring. Also, we studied the relation between strongly nil regular clean and nil regular clean rings, unit regular rings, ɱ-regular rings and strongly π −regular rings.

Keywords: Clean rings, strongly Nil Regular Clean rings, strongly π −regular rings

1 Introduction

Throughout this paper, Ꞧ denotes associative rings with identity; for a subset χ of Ꞧ, the right (lift) annihilator of χ is denoted by (r(χ)) and (l(χ)). If χ={ƋƋ}, we usually abbreviate it to r(ƌ), l(ƌ). A ring Ꞧ is said to be von Neumann regular ring (or just regular) if and only if for each ƌ in Ꞧ, there exists b in Ꞧ such that ƌ=ƌbƌ [1]. McCoy in [1], called a ring Ꞧ is ɱ-regular ring if for each ƌ ∈ Ꞧ there exist a positive fixed number ɱ such that $ƌ^{ɱ} = ƌ^{ɱ} b ƌ^{ɱ}$ is regular for some $b \in$ Ꞧ. A ring Ꞧ is said to be strongly π −regular, if for all a in Ꞧ there exist a positive integer n such that $ƌ^{n} = ƌ^{n+1} b$ [2]. Ꞧ is called local ring, if it has exactly one maximal ideal [3], [4]. Ring Ꞧ is called clean, if each element ƌ in Ꞧ can be written as, $ƌ = ҿ + u$, where ҿ is an idempotent element and u is unit element [5]. Hani in [6], called a ring Ꞧ as a Nil Regular Clean or (ƝꞦ-Clean) if for each element ƌ in Ꞧ, there exits $ꞵ \in reg(Ꞧ)$ and $ɲ \in Ɲ(Ꞧ)$ such that $ƌ = ꞵ + ɲ$, ling and long in [7], define a ring to be a UꞦ-ring if every element is a sum of a unit and a regular. A ring Ꞧ is called strongly Nil-clean, if for each element ƌ in Ꞧ can be written as a sum of an idempotent and a nilpotent (that commute with each other) [8] [9] [10].

For a ring Ʀ, we will use U̇(Ʀ), Ɲ(Ʀ), reg(Ʀ), Ĵ(Ʀ) and Id(Ʀ) to denote the group of units, the set of nilpotent elements, the set of regular elements, the Jacobson radical and the set of idempotent elements of Ʀ, respectively.

2 The Strongly Nil Regular Clean Rings

In this section we give the definition of strongly Nil Regular Clean with some of its characterization and basic properties.

Definition 2.1: An element ƌ in Ʀ is said to be strongly Nil Regular Clean or (strongly ƝƦ-Clean element), if it is the sum of a (Von Neumann) regular and a nilpotent (that commute with each other), Ʀ is called strongly Nil Regular Clean ring if every element of Ʀ is strongly Nil Regular Clean.

Example: Z_6 is strongly ƝƦ-clean ring.

Proposition 2.2: If Ʀ is strongly $ƝƦ - Clean$ ring, then $ƌ - ƌ^2$is $ƝƦ - Clean$ element for all ƌ in Ʀ.

Proof: Let $ɼ \in reg(Ʀ)$, $ɳ \in Ɲ(Ʀ)$, s.t $ƌ = ɳ + ɼ$, $ƌ^2 = (ɳ + ɼ)^2 = ɼ^2 + 2ɳɼ + ɳ^2$. Hence, $ƌ - ƌ^2 = ɼ + ɳ - ɼ^2 - (2ɼɳ + ɳ^2)$, $ƌ - ƌ^2 = (ɼ - ɼ^2) + (1 - 2ɼ - ɳ)ɳ$, $ɼ - ɼ^2 \in reg(Ʀ)$, $(1 - 2ɼ - ɳ)ɳ \in Ɲ(Ʀ)$. Therefore, $ƌ - ƌ^2$ is $ƝƦ - Clean$ element. #

Corollary 2.3: If Ʀ is strongly $ƝƦ - Clean$ ring with only [0,1] as regular element, then $ƌ - ƌ^2$ is nilpotent element.

Proof: by proposition 2.2 $ƌ - ƌ^2$ is $ƝƦ - Clean$ element that is $ƌ - ƌ^2 = ɼ - ɼ^2 + (1 - 2ɼ - ɳ)ɳ = ɼ(1 - ɼ) + (1 - 2ɼ - ɳ)ɳ$. Now, since $ɼ \in [0,1]$, if $ɼ = 0$, or $ɼ = 1$, hence $ɼ(1 - ɼ) = 0$. Therefore, $ƌ - ƌ^2 = (1 - 2ɼ - ɳ)ɳ \in Ɲ(Ʀ)$. #

Proposition.2.4: A ring Ʀ is strongly $ƝƦ - Clean$ if and only if $Ʀ\,/\,Ĵ(Ʀ)$ is Ɓoolean and $Ĵ(Ʀ)$ is nil, and Ʀ with only [0,1] as regular element.

Proof: Let $ƌ \in Ʀ$, then $ƌ - ƌ^2 \in Ĵ(Ʀ)$ by corollary(2.3), as $Ĵ(Ʀ)$ is nil, there exists $ɼ \in reg(Ʀ)$, $ƌ - ɼ \in Ĵ(Ʀ)$, hence $ƌ = ɼ + (ƌ - ɼ)$ is a strongly $ƝƦ - clean$ decomposition. Assume that Ʀ is strongly $ƝƦ - clean$ ring so $Ʀ\,/\,Ĵ(Ʀ)$ is Ɓoolean by [6, proposition.2.23]. Now let $ƌ = ɼ + ɳ$, $ɼɳ = ɳɼ$, be strongly $ƝƦ - clean$ decomposition, as $Ʀ\,/\,Ĵ(Ʀ)$ is Ɓoolean and ɳ is nilpotent, it follows that $ɳ \in Ĵ(Ʀ)$ thus $ɼ = ƌ - ɳ \in Ĵ(Ʀ)$, so $ɼ = 0$. Therefore $ƌ = ɳ$, is a nilpotent. #

Proposition.2.5: An element $ƌ \in Ʀ$ is strongly $ƝƦ - Clean$, if ƌ is ÛƦ −ring and $ƌ - ƌ^2$ is $ƝƦ - Clean$.

Proof: Let us $ꝺ = ꞅ + u = (1 - ꞅ) + (-1 + 2ꞅ + u)$, be $\hat{U}Ꞧ -$ring decomposition in $Ꞧ$, and $ꝺ - ꝺ^2$ be $ƝꞦ - Clean$. $ꝺ^2 = (ꞅ + u)^2 = ꞅ^2 + 2ꞅu + u^2$, and so, $ꝺ - ꝺ^2 = (ꞅ + u) - (ꞅ^2 + 2ꞅu + u^2) = (ꞅ - ꞅ^2) + (u - 2ꞅu - u^2) = ꞅ(1 - ꞅ) + (1 - 2ꞅ - u)u$, that is $(1 - 2ꞅ - u)$ is nilpotent. So $ꝺ = (1 - ꞅ) + (-1 + 2ꞅ + u)$ is strongly $ƝꞦ - Clean$ decomposition in $Ꞧ$. #

Proposition 2.6: Let $Ꞧ$ be a strongly nil clean ring. Then $Ꞧ$ is strongly $ƝꞦ - Clean$ ring and $ꝺ - ꝺ^2$ is nilpotent.

Proof: Let $ꝺ$ be strongly nil clean $ꝺ = ȩ + ꞑ$, $ꞑ \in Ɲ(Ꞧ)$ and since $ȩ \in idem(Ꞧ)$, $-ꝺ = -ȩ - ꞑ$, $-ȩ \in reg(Ꞧ), -ꞑ \in Ɲ(Ꞧ)$, therefore $ꝺ = ꞅ + ꞑ$ which is $ƝꞦ - Clean$. $ꝺ = ȩ + ꞑ, ꝺ^2 = ȩ + 2ȩꞑ + ꞑ^2$, $ꝺ - ꝺ^2 = ȩ + ꞑ - (ȩ + 2ȩꞑ + ȩ + 2ȩꞑ + ꞑ^2) = ꞑ - 2ȩꞑ - ꞑ^2 = (1 - 2ȩ - ꞑ)ꞑ \in Ɲ(Ꞧ)$. #

Proposition 2.7: Let $Ꞧ$ be an $ɱ - regular$ abelian ring. Then any element $ꝺ$ in $Ꞧ$ is strongly $ƝꞦ$-Clean if there exist $ꞅ \in reg(Ꞧ)$, such that., $ꝺ - ꞅ$, and every non zero divisor is nilpotent.

Proof: Since $Ꞧ$ is $ɱ - regular$, there exist $b \in Ꞧ$, and a fixed positive integer $ɱ$ such that, $ꝺ^ɱ = ꝺ^ɱ b ꝺ^ɱ$, and we have by [11, proposition.2.1.12] $ꞅ(ꝺ^ɱ b) \cap ꝺ^ɱ Ꞧ = 0$, and there exists $0 \neq x \in ꞅ(ꝺ^ɱ b) \cap ꝺ^ɱ Ꞧ$, so we get $ꝺ^ɱ bx = 0$, $\chi = ꝺ^ɱ ꞅ = ꝺ^ɱ b ꝺ^ɱ r = ꝺ^ɱ bx = 0$, but $ꞅ(ꝺ^ɱ b)$ is essential that is, $ꝺ^ɱ Ꞧ = 0$, $ꝺ^ɱ = 0$, $\chi = 0$. Now, to prove that $ꝺ - ꞅ$, is a non-zero divisor, let $\chi \in Ꞧ$, s.t. $(ꝺ - ꞅ)\chi = 0$, that is $ꝺ\chi = ꞅ\chi$, but $ꞅ\chi \in ꞅꞦ = ꞅ(ꝺ)$, then $ꝺꞅ\chi = 0$, but $ꝺ\chi \in ꝺꞦ$ and $ꞅ\chi \in ꞅꞦ$, then $ꝺ\chi = ꞅ\chi \in ꝺꞦ \cap ꞅꞦ = 0$, so $\chi \in ꞅ(ꝺ) = ꞅꞦ$, therefore $ꝺ\chi$=0, χ=0. Hence $ꝺ - ꞅ$ is a non-zero divisor, then by hypothesis $ꝺ - ꞅ$ is nilpotent so, $ꝺ - ꞅ$=$ꞑ$, $ꞑ \in Ɲ(Ꞧ)$. Therefore $ꝺ = ꞑ + ꞅ$, #

Proposition 2.8: Let $Ꞧ$ be CI ring, Then $Ꞧ$ is strongly $\pi - regular$ if and only if it is strongly $ƝꞦ - Clean$ ring.

Proof: Suppose that $Ꞧ$ is strongly $\pi - regular$ then $Ꞧ$ is strongly $ƝꞦ - Clean$ ring, by [theorem 8.3 in 12] we have $ꝺ = ȩu + w$, $w \in Ɲ(Ꞧ)$, $ȩu = ꞅ \in reg(Ꞧ)$. Now, assume $Ꞧ$ is strongly $ƝꞦ - Clean$ ring, $ꝺ = ꞅ + ꞑ$, $ꞅꞑ = ꞑꞅ$, $ꞑ \in Ɲ(Ꞧ), ꞅ \in reg(Ꞧ)$, and by [12, proposition 5.1.1], $ȩu$ is strongly regular so, $ȩu = ꞅ$ and $ꝺ = ȩu + ꞑ$. Therefore by [12, theorem 8.3] $Ꞧ$ is strongly $\pi - regular$ and u $ȩ$ and $ꞑ$ are commutative with each other. #

Proposition 2.9: Let $Ꞧ$ be an abelian local ring with every zero-divisor is strongly $\pi - regular$, then $Ꞧ$ is strongly $ƝꞦ - clean$ ring.

Proof: Let $Ꞧ$ be a ring with every zero-divisor is strongly $\pi - regular$, and let $ꝺ \epsilon Ꞧ$, if $ꝺ \epsilon reg(Ꞧ)$, then $ꝺ = ꝺ + 0$ that is $ꝺ$ is a sum of regular and 0 (nilpotent). Now, if $ꝺ \notin reg(Ꞧ)$, by hypothesis, $ꝺ \epsilon Ꞧ$ is strongly $\pi - regular$, then there exists $n \in N$ $b \in R$ such that $a^n = a^{n+1} b$, $b = b^2 a$, let $c = b^n a^n b^n$, then $a^n = a^{2n} c$, and $c = c^2 a^n$, set $v = c + 1 - a^n c$, then $v^{-1} = a^n + 1 - a^n c$, set $f = a^n c$, then $f^2 = a^{2n} c^2 = (a^{2n} c)c = a^n c = f \in R$, choose $w = v^{-1}$ then $a^n = fw$, let $r = 1 - f$ and $u = r - a + 1$ choose $z = a^{n-1} w^{-1} f - (1 + a + \cdots + a^{n-1}) r$ then $uz = (r - a + 1)(a^{n-1} w^{-1} f - (1 + a + \cdots + a^{n-1}) r) = a^n w^{-1} f + (1 - a)(1 + a + \cdots + a^{n-1}) r = f + (1 - a^n) r = f + r = 1$. Like wise $zu = 1$, then $a = r + (1 - u)$, where $r = r^2$, and $u \in U(R)$ since R is local then $(1 - u) \in N(R)$. Therefore R is strongly $NR - clean$ ring. #

Acknowledgements. The author is grateful to the University of Mosul, College of Computer Science and Mathematics.

References

[1] N. H. McCoy, Generlized regular rings, *Bull. Amer. Math*. Soc., 45 (2) (1939), 175-178. https://doi.org/10.1090/s0002-9904-1939-06933-4

[2] G. Azumaya, Strongly π −regular rings, *J. Fac. Sci. Hokkaido University*, **13** (1954), 34-39. https://doi.org/10.14492/hokmj/1530842562

[3] D. M. Burton, *Introduction to modern abstract algebra*, Addison-Wesley pub. Co., 1967.

[4] M. Hazewinkel, N. Gubareni and V. V. Kirichenko, *Algebra, rings and modules*, Vol.1, Springer science & Business Media, 2004. https://doi.org/10.1007/978-1-4020-5141-8

[5] N. Ashrafi and E. Nasibi, r-clean rings, ArXiv prepr.arxiv1104.2167, 20011.

[6] Hani Khashan, NR-Clean Rings, *Vietnam Journal of Mathematics* (2016). https://doi.org/10.1007/s10013-016-0197-8

[7] Y. Z. Ling and C. J. Long, On UR-rings, *J. of Math. Research and Exposition*, **29** (2) (2009), 355-361. https://doi.org/10.4153/cjm-1956-040-9

[8] A. J. Diesl, Alexander J., Nil clean rings, *Journal of Algebra*, 383 (2013), 197-211. https://doi.org/10.1016/j.jalgebra.2013.02.020

[9] T. Kosan, T. K. Lee, Y, Zhou, When is every matrix over a division ring a sum of an idempotent and a nilpotent, *Linear Algebra Appl.*, **450** (2014), 7-12. https://doi.org/10.1016/j.laa.2014.02.047

[10] S. Breaz, G. Călugăreanu, P. Danchev, T. Micu, Nil-clean matrix rings, *Linear Algebra Appl*., **439** (2013), 3115-3119. https://doi.org/10.1016/j.laa.2013.08.027

[11] M. F. Khalf, *On η-Regular Rings*, M. S Thesis, Mosul University, 2011.

[12] M. R. M, Al-Kouri, *On Strongly π-Regular Rings*, Ph.D. Thesis, Mosul University, 2000.

Received: May 21, 2021; Published: June 11, 2021