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Prime Counting Function in Base of $\frac{x}{3}$

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Abstract

In this study, we present the function $H(x)_p$ based on $P_k(x,a)$ introduced by Lehmer. $H(x)_p$ denotes the number of numbers that are not divisible by prime numbers $< p$ but are divisible by p. Herein, we show that $H(x)_p$ can be obtained only using $\frac{x}{3}$. We also present our own prime counting function based on $H(x)_p$, that is, $\frac{x}{3}$.

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1 Introduction

A prime counting function represents the number of primes below a certain limit. $P_k(x,a)$ denotes the number of products $\leq x$ of k primes, each greater than p_a. Therefore, the difference between $H(x)_p$ and $P_k(x,a)$ is that $P_k(x,a)$ only takes primes greater than p_a while $H(x)_p$ takes primes greater than or equal to p_a, where p_a is required to be in the product [1–2].
p_s represents the prime in the position s.
For every natural number a, prime number c, and composite number b, we have the following definitions:

Definitions.

$$H(x)_p \coloneqq \#[\, ap_s \leq x | p < p_s \dagger a] \tag{1}$$

$$R_s\left(\frac{x}{p_s}\right) \coloneqq \#[\ cp_s \leq x | p_s \leq c]. \tag{2}$$

This indicates that $R_s\left(\frac{x}{p_s}\right) = \pi\left(\frac{x}{p_s}\right) - s + 1$

$$T_p(\frac{x}{p_s}) \coloneqq \#[\ bp_s \leq x | p < p_s \dagger b]. \tag{3}$$

Form the previous definitions,

$$H(x)_p = R_p\left(\frac{x}{p_s}\right) + T_p\left(\frac{x}{p_s}\right) + 1. \tag{4}$$

Therefore, $H(x)_p$ satisfices

$$\mathbf{1} + \sum_{\boldsymbol{S=1}} H(x)_p = \boldsymbol{x}. \tag{5}$$

In the next chapter, we present $I(x)$, where, for x real,

$$I(x) \coloneqq \mathbf{1} + \sum_{\boldsymbol{S=3}} H(x)_p. \tag{6}$$

2 Obtaining $\boldsymbol{I(x)}$

Lemma 2.1 From equation 6 and taking $3|x$,

$$\frac{x}{3} = I(x). \tag{7}$$

Proof:
For $2|x$,

$$H(x)_2 = \frac{x}{2} \tag{8}$$

and

$H(x)_3 = \frac{x}{3} - \frac{x}{2*3}$. Therefore,

$$H(x)_3 = \frac{x}{6}. \tag{9}$$

For $6|x$, we obtain,

$x - H(x)_2 - H(x)_3 = x - \frac{x}{2} - \frac{x}{6} = \frac{x}{3}$.

Theorem 2. 1 From lemma 2.1 and x as a positive natural number,

$$I(x) = \begin{cases} \left\|\frac{x+1}{3}\right\| \; if\ 3 \dagger x\ and\ 3|x+1 \\ \left\|\frac{x+2}{3}\right\| \; if\ 2{,}3 \dagger x\ and\ 3|x+2. \\ \left\|\frac{x}{3}\right\| \; otherwise \end{cases} \tag{10}$$

Proof:
Consider a natural number c.
Every number $\frac{c}{3}$, where $c \in I^{-1}(x)$ is between $\frac{x}{3}\ and\ \frac{x+3}{3}$; therefore, 1 or 2 must be added to c such that the result is the next number divisible by 3. For example, $\frac{c+1}{3} \in I(x)$ or $\frac{c+2}{3} \in I(x)$.

3 Functions involving $I(x)$

Here, we introduce T(x), which defines the number of composite numbers that are not divisible by 2 and 3 up to x, and $R(x)$, where $R(x) = \pi(x) - 2$ for $3 \leq x$.
Therefore,

$$\boldsymbol{R(x) + T(x) + 1 = I(x).} \tag{11}$$

Lemma 3.1 From equations 2 and 3 and the definition of $T(x)$,

$$T(x) = \sum_{S=3} R_p\left(\frac{x}{p_s}\right) + T_p\left(\frac{x}{p_s}\right). \tag{12}$$

Proof:

$$\mathbf{1} + \left(\sum_{\boldsymbol{S=1}} H(x)_p\right) - H(x)_2 - H(x)_3 =$$

$$1 + \left(\sum_{S=1} R_p\left(\frac{x}{p_s}\right) + T_p\left(\frac{x}{p_s}\right) + 1\right) - \left(R_2\left(\frac{x}{p_1}\right) + T_2\left(\frac{x}{p_1}\right) + 1\right)$$

$$-\left(R_3\left(\frac{x}{p_2}\right) + T_3\left(\frac{x}{p_2}\right) + 1\right) =$$

$1 + \left(\sum_{S=3} 1\right) + \sum_{S=3} R_p\left(\frac{x}{p_s}\right) + T_p\left(\frac{x}{p_s}\right)$,

where

$$\left(\sum_{S=3} 1\right) = R(x)$$

and

$T(x) = \sum_{S=3} R_s\left(\frac{x}{p_s}\right) + T_p\left(\frac{x}{p_s}\right)$.

Lemma 3.2 Consider equation 2 and the definition of $R(x)$. Then,

$$R\left(\frac{x}{p_s}\right) - R_p\left(\frac{x}{p_s}\right) = s - 3. \tag{13}$$

Proof:
We know that $R_p\left(\frac{x}{p_s}\right) = \boldsymbol{\pi}\left(\frac{x}{p_s}\right) - s + 1$ and $\boldsymbol{R}\left(\frac{x}{p_s}\right) = \boldsymbol{\pi}\left(\frac{x}{p_s}\right) - \mathbf{2}$
$R\left(\frac{x}{p_s}\right) - R_p\left(\frac{x}{p_s}\right) = \left(\boldsymbol{\pi}\left(\frac{x}{p_s}\right) - \mathbf{2}\right) - \left(\boldsymbol{\pi}\left(\frac{x}{p_s}\right) - s + 1\right) = s - 3.$

Lemma 3.3 Consider equations 12 and 3. The difference between them is

$$T\left(\frac{x}{p_s}\right) - T_p\left(\frac{x}{p_s}\right) = \sum_{i=3}^{s-1} H\left(\frac{x}{p_k}\right)_p - 1. \tag{14}$$

Proof:
In this case, p_s is constant for every prime p_k, where $3 \leq k \leq s - 1$.

$$T\left(\frac{x}{p_s}\right) = \sum_{i=3}^{k} R_p\left(\frac{x}{p_k p_s}\right) + T_p\left(\frac{x}{p_k p_s}\right) \tag{15}$$

Equation 15 shows that, for all the primes p_k, we obtain numbers divisible by primes $< p_s$; therefore, to obtain $T_p\left(\frac{x}{p_s}\right)$, we must eliminate all those numbers, meaning
$T\left(\frac{x}{p_s}\right) - \sum_{i=3}^{s-1} H\left(\frac{x}{p_k}\right)_p - 1 = T_p\left(\frac{x}{p_s}\right).$

From lemmas 3.1, 3.2, and 3.3, we obtain

$$T(x) = \sum_{i=3}^{s} \left(R\left(\frac{x}{p_s}\right) + T\left(\frac{x}{p_s}\right) - s + 3 - d\right) \tag{16}$$

and $\sum_{i=3}^{s-1} H\left(\frac{x}{p_k}\right)_p - 1 = d$, for a reduction in the computing.

4 $\frac{x}{3}$ and $\pi(x)$

Theorem 4.1
For a natural x,

$$\boldsymbol{T}(\boldsymbol{x}) = \sum_{i=3}^{s}\left(\left(I\left(\frac{x}{p_s}\right) - 1\right) - s + 3 - d\right). \tag{17}$$

Proof:
From equation 11, we replace $R(x) + T(x)$ for $I(x) - 1$ and obtain equation 17.

Then, by reducing the equation, we obtain

$$\left(I\left(\frac{x}{p_{sk}}\right)-1\right)-s_k+3 \coloneqq \sum_{i=3}\left(\left(I\left(\frac{x}{p_s}\right)-1\right)-s+3-d\right), \tag{18}$$

where the subscripts represent positions. For example, if we have p_4 and $H\left(\frac{x}{p_4 p_3}\right)_5-1$, which we must eliminate in relation to p_4. Then

$$H\left(\frac{x}{p_4 p_3}\right)_5-1 \coloneqq \left(I\left(\frac{x}{p_{43}}\right)-1\right)-s_3+3.$$

Theorem 4.2 The prime counting function $\pi(x)$ is given by

$$\boldsymbol{\pi(x)}=\boldsymbol{I(x)}-\left(\left(I\left(\frac{x}{p_{sk}}\right)-1\right)-s_k+3\right)+1. \tag{19}$$

Proof:
From equation 11 and the definition of $R(x)$,
$\pi(x)=I(x)-T(x)+1$.
Therefore, from theorem 4.1, we obtain theorem 4.2.

Example:

$$x=100$$

$$\boldsymbol{I(100)}=\left\|\frac{\mathbf{99+1}}{\mathbf{3}}\right\|=\frac{\mathbf{99}}{\mathbf{3}}=\mathbf{33}$$

$$\boldsymbol{T(100)}=\left(\left(I\left(\frac{x}{p_3}\right)-1\right)-3+3\right)+\left(I\left(\frac{x}{p_4}\right)-1\right)-4+3=$$

$$\left(\left(I(20)-1\right)\right)+\left(I(14)-1\right)-1=\frac{19+2}{3}+\frac{14+1}{3}-3=9$$

$\boldsymbol{\pi(100)}=\mathbf{33-9+1=25.}$

Note:
In $I(x)-T(x)$, all the composite numbers are eliminated from a set of numbers, which is the same as the Eratosthenes algorithm. Thus, when we use $T(x)$, we obtain the Eratosthenes algorithm. [3]

Conclusion

We have shown a prime number counting function and presented the function $H(x)_p$ that has a simple relation with $\frac{x}{3}$.

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