Prime Counting Function in Base of $\frac{x}{3}$

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Abstract

In this study, we present the function $H(x)_p$ based on $P_k(x, a)$ introduced by Lehmer. $H(x)_p$ denotes the number of numbers that are not divisible by prime numbers $< p$ but are divisible by $p$. Herein, we show that $H(x)_p$ can be obtained only using $\frac{x}{3}$. We also present our own prime counting function based on $H(x)_p$, that is, $\frac{x}{3}$.

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1 Introduction

A prime counting function represents the number of primes below a certain limit. $P_k(x, a)$ denotes the number of products $\leq x$ of $k$ primes, each greater than $p_a$. Therefore, the difference between $H(x)_p$ and $P_k(x, a)$ is that $P_k(x, a)$ only takes primes greater than $p_a$ while $H(x)_p$ takes primes greater than or equal to $p_a$, where $p_a$ is required to be in the product $[1–2]$. $p_s$ represents the prime in the position $s$.

For every natural number $a$, prime number $c$, and composite number $b$, we have the following definitions:

Definitions.

\[ H(x)_p := \# \{ ap_s \leq x | p < p_s \uparrow a \} \] (1)
\[ R_s \left( \frac{x}{p_s} \right) \coloneqq \# \left[ cp_s \leq x | p_s \leq c \right]. \]  \hspace{1cm} (2)

This indicates that \( R_s \left( \frac{x}{p_s} \right) = \pi \left( \frac{x}{p_s} \right) - s + 1 \)

\[ T_p \left( \frac{x}{p_s} \right) \coloneqq \# \left[ bp_s \leq x | p < p_s \dagger b \right]. \]  \hspace{1cm} (3)

Form the previous definitions,

\[ H(x)_p = R_p \left( \frac{x}{p_s} \right) + T_p \left( \frac{x}{p_s} \right) + 1. \]  \hspace{1cm} (4)

Therefore, \( H(x)_p \) satisifies

\[ 1 + \sum_{s=1}^{m} H(x)_p = x. \]  \hspace{1cm} (5)

In the next chapter, we present \( I(x) \), where, for \( x \) real,

\[ I(x) := 1 + \sum_{s=3}^{x} H(x)_p. \]  \hspace{1cm} (6)

2 Obtaining \( I(x) \)

**Lemma 2.1** From equation 6 and taking \( 3|x \),

\[ \frac{x}{3} = I(x). \]  \hspace{1cm} (7)

**Proof:**

For \( 2|x \),

\[ H(x)_2 = \frac{x}{2} \]  \hspace{1cm} (8)

and

\[ H(x)_3 = \frac{x}{3} - \frac{x}{2+3}. \]  \hspace{1cm} (9)

For \( 6|x \), we obtain,

\[ x - H(x)_2 - H(x)_3 = x - \frac{x}{2} - \frac{x}{6} = \frac{x}{3}. \]

**Theorem 2.1** From lemma 2.1 and \( x \) as a positive natural number,
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\[
I(x) = \begin{cases} 
\left\lceil \frac{x+1}{3} \right\rceil & \text{if } 3 \nmid x \text{ and } 3 \mid x + 1 \\
\left\lceil \frac{x+2}{3} \right\rceil & \text{if } 2, 3 \mid x \text{ and } 3 \mid x + 2. \\
\left\lceil \frac{x}{3} \right\rceil & \text{otherwise}
\end{cases}
\tag{10}
\]

**Proof:**
Consider a natural number \( c \).
Every number \( c \), where \( c \in I^{-1}(x) \), is between \( \frac{x}{3} \) and \( \frac{x+3}{3} \); therefore, 1 or 2 must be added to \( c \) such that the result is the next number divisible by 3. For example, \( \frac{c+1}{3} \in I(x) \) or \( \frac{c+2}{3} \in I(x) \).

### 3 Functions involving \( I(x) \)

Here, we introduce \( T(x) \), which defines the number of composite numbers that are not divisible by 2 and 3 up to \( x \), and \( R(x) \), where \( R(x) = \pi(x) - 2 \) for \( 3 \leq x \).

Therefore,

\[
R(x) + T(x) + 1 = I(x). \tag{11}
\]

**Lemma 3.1** From equations 2 and 3 and the definition of \( T(x) \),

\[
T(x) = \sum_{s=1}^{\pi} R_p \left( \frac{x}{p_s} \right) + T_p \left( \frac{x}{p_s} \right). \tag{12}
\]

**Proof:**

\[
1 + \left( \sum_{s=1}^{\pi} H(x)p_s \right) - H(x)_2 - H(x)_3 = 1 + \left( \sum_{s=1}^{\pi} R_p \left( \frac{x}{p_s} \right) + T_p \left( \frac{x}{p_s} \right) + 1 \right) - \left( R_2 \left( \frac{x}{p_1} \right) + T_2 \left( \frac{x}{p_1} \right) + 1 \right) - \left( R_3 \left( \frac{x}{p_2} \right) + T_3 \left( \frac{x}{p_2} \right) + 1 \right) = 1 + \sum_{s=3}^{\pi} \sum_{p_s=1}^{\pi} R_p \left( \frac{x}{p_s} \right) + T_p \left( \frac{x}{p_s} \right),
\]

where

\[
\sum_{s=3}^{\pi} 1 = R(x)
\]

and

\[
T(x) = \sum_{s=3}^{\pi} R_s \left( \frac{x}{p_s} \right) + T_p \left( \frac{x}{p_s} \right).
\]

**Lemma 3.2** Consider equation 2 and the definition of \( R(x) \). Then,
\[ R \left( \frac{x}{p_s} \right) - R_p \left( \frac{x}{p_s} \right) = s - 3. \]  (13)

**Proof:**
We know that \( R_p \left( \frac{x}{p_s} \right) = \pi \left( \frac{x}{p_s} \right) - s + 1 \) and \( R \left( \frac{x}{p_s} \right) = \pi \left( \frac{x}{p_s} \right) - 2 \)

\[ R \left( \frac{x}{p_s} \right) - R_p \left( \frac{x}{p_s} \right) = \left( \pi \left( \frac{x}{p_s} \right) - 2 \right) - \left( \pi \left( \frac{x}{p_s} \right) - s + 1 \right) = s - 3. \]

**Lemma 3.3** Consider equations 12 and 3. The difference between them is

\[ T \left( \frac{x}{p_s} \right) - T_p \left( \frac{x}{p_s} \right) = \sum_{i=3}^{s-1} H \left( \frac{x}{p_k} \right)_p - 1. \]  (14)

**Proof:**
In this case, \( p_s \) is constant for every prime \( p_k \), where \( 3 \leq k \leq s - 1 \).

\[ T \left( \frac{x}{p_s} \right) = \sum_{i=3}^{k} R_p \left( \frac{x}{p_k p_s} \right) + T_p \left( \frac{x}{p_k p_s} \right) \]  (15)

Equation 15 shows that, for all the primes \( p_k \), we obtain numbers divisible by primes \( < p_s \); therefore, to obtain \( T_p \left( \frac{x}{p_s} \right) \), we must eliminate all those numbers, meaning

\[ T \left( \frac{x}{p_s} \right) - \sum_{i=3}^{s-1} H \left( \frac{x}{p_k} \right)_p - 1 = T_p \left( \frac{x}{p_s} \right). \]

From lemmas 3.1, 3.2, and 3.3, we obtain

\[ T(x) = \sum_{i=3}^{s} \left( R \left( \frac{x}{p_s} \right) + T \left( \frac{x}{p_s} \right) - s + 3 - d \right) \]  (16)

and \( \sum_{i=3}^{s-1} H \left( \frac{x}{p_k} \right)_p - 1 = d \), for a reduction in the computing.

**4 \( \frac{x}{3} \) and \( \pi(x) \)**

**Theorem 4.1**
For a natural \( x \),

\[ T(x) = \sum_{i=3}^{s} \left( I \left( \frac{x}{p_s} \right) - 1 \right) - s + 3 - d \].  (17)

**Proof:**
From equation 11, we replace \( R(x) + T(x) \) for \( I(x) - 1 \) and obtain equation 17.
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Then, by reducing the equation, we obtain

$$
(I\left(\frac{x}{p_{sh}}\right) - 1) - s_k + 3 := \sum_{i=3}^{\infty} \left( I\left(\frac{x}{p_i}\right) - 1 \right) - s + 3 - d,
$$

where the subscripts represent positions. For example, if we have $p_4$ and $H\left(\frac{x}{p_4p_3}\right)_5 - 1$, which we must eliminate in relation to $p_4$. Then

$$
H\left(\frac{x}{p_4p_3}\right)_5 - 1 := (I\left(\frac{x}{p_4}\right) - 1) - s_3 + 3.
$$

**Theorem 4.2** The prime counting function $\pi(x)$ is given by

$$
\pi(x) = I(x) - \left( I\left(\frac{x}{p_{sk}}\right) - 1 \right) - s_k + 3 + 1.
$$

**Proof:**

From equation 11 and the definition of $R(x)$,

$$
\pi(x) = I(x) - T(x) + 1.
$$

Therefore, from theorem 4.1, we obtain theorem 4.2.

**Example:**

$$
I(100) = \left\| \frac{99 + 1}{3} \right\| = \frac{99}{3} = 33
$$

$$
T(100) = \left( (I\left(\frac{x}{p_3}\right) - 1) - 3 + 3 \right) + \left( I\left(\frac{x}{p_4}\right) - 1 \right) - 4 + 3 = (I(20) - 1) + (I(14) - 1) - 1 = \frac{19 + 2}{3} + \frac{14 + 1}{3} - 3 = 9
$$

$$
\pi(100) = 33 - 9 + 1 = 25.
$$

**Note:**

In $I(x) - T(x)$, all the composite numbers are eliminated from a set of numbers, which is the same as the Eratosthenes algorithm. Thus, when we use $T(x)$, we obtain the Eratosthenes algorithm. [3]

**Conclusion**

We have shown a prime number counting function and presented the function $H(x)_p$ that has a simple relation with $\frac{x}{3}$.
References


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