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Tensor Product of Order Ideals

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Abstract

In this paper, it is shown that tensor product of order ideals is an order ideal.

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1 Introduction

In [6], D. H. Fremlin presented the tensor product of Archimedean vector lattices. After that, many authors studied this subject , for example Grobler , Amor, Buskes and so on.In [8] Grobler et al introduced the tensor product of Archimedean ordered vector spaces.In [2], Amor showed that tensor product of two d- algebras is a d-algebra. In [3], Azouzi et al proved that the Riesz tensor product of f-algebras is an f-algebra. For d-algebras, we can give a reference in [4]. In [5], Buskes et al proved that the tensor product of f-algebras is an f-algebra by different method. In this work , we prove that the Riesz tensor product of order ideals is also an order ideal.

For unexplained notion and terminology, we refer to the books [1, 10].

2 Preliminary Notes

An ordered real vector space E with the property that for every $x, y \in E$ the supremum and infimum of $\{x, y\}$ exist in E is called a Riesz space or a

vector lattice. We denote the following notations for supremum and infimum: $x \vee y = sup\{x, y\}$ and $x \wedge y = inf\{x, y\}$.The absolute value or modulus of $x \in E$ is given by the formula : $|x| = x \vee -x$. The set of all positive elements in a vector lattice E is said to be the positive cone and denoted by E^+. So, $E^+ = \{x \in E : x \geq 0\}$. A vector lattice E is called Archimedean if $\frac{x}{n} \downarrow 0$ holds in E for every $x \in E^+$. In this paper, we will assume that all vector lattices are Archimedean. We say that a vector lattice E is Dedekind complete if every subset of E which is bounded from above has a supremum. Let E be a vector lattice and let $x, y \in E$. The set $[x, y] = \{z \in E : x \leq z \leq y\}$ is said to be an order interval. A subset A of E is called an order bounded set if there exist $x, y \in E$ such that $A \subseteq [x, y]$. Let E be a vector lattice. A subset A of a vector lattice E is called solid whenever $|x| \leq |y|$ and $y \in A$ imply $x \in A$. A solid vector subspace of a vector lattice is said to be an order ideal.A Riesz subspace (or vector sublattice) G of a Riesz space E is said to be order dense in E whenever for each $0 < x \in E$ there exists some $y \in G$ with $0 < y \leq x$. A net $\{x_\alpha\}$ of a vector lattice E is called order convergent to a vector x whenever there exists a net $\{y_\alpha\}$ satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all α. A subset A of a vector lattice E is called order closed whenever $\{x_\alpha\}$ in A and $\{x_\alpha\}$ is order convergent to x imply $x \in A$. An order closed ideal is called a band. Let A be a non-empty subset of a vector lattice E.Then the ideal generated by A is the smallest ideal (with respect to inclusion) that includes A. That is, this ideal is

$$E_A = \{x \in E : there \quad exist \quad x_1, ..., x_n \in A, 0 < \lambda \in R \quad with \quad |x| \leq \lambda \Sigma_{i=1}^n |x_i|\}$$

The ideal generated by a vector $x \in E$ is called a principal ideal and denoted by E_x. It is dedfined by the set

$$E_x = \{y \in E : there \quad exists \quad 0 < \lambda \quad with \quad |y| \leq \lambda |x|\}.$$

A sequence (x_n) in a vector lattice is said to be uniformly Cauchy whenever there exists some $0 < u$ such that for each $\varepsilon > 0$ we have $|x_n - x_m| \leq \varepsilon u$ for all n and m sufficiently large. A sequence (x_n) in a vector lattice is called relatively uniformly convergent to x whenever there exist some $u > 0$ and a sequence (r_n) of real numbers with $r_n \downarrow 0$ such that $|x_n - x| \leq r_n u$ holds for all n.

A vector lattice is called uniformly complete whenever every uniformly Cauchy sequence is relatively uniformly convergent. A norm $\|.\|$ on a vector lattice E is called a lattice norm if $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$ for all $x, y \in E$. Then, $(E, \|.\|)$ is called a normed vector lattice. A normed complete vector lattice E is called a Banach lattice.A lattice norm $\|.\|$ on E is called an M-norm if $\|x \vee y\| = max\{\|x\|, \|y\|\}$ for all $x, y \in E^+$. An M-normed Banach lattice E is called an M-space or AM-space.

Theorem 2.1 [1, 10] *Every $x \in E^+$ is an order unit of*

$$E_x = \{y : |y| \leq \lambda |x| for \quad some \quad \lambda > 0\}$$

is an M-norm on E. This norm is called the order unit norm. If E_x is uniformly complete, then $(E_x, \|.\|)$ is an M-space. Therefore it is a Banach lattice.

Theorem 2.2 [1, 10] *(Kakutani-Bohnenblust-M.Krein-S. Krein) A Banach lattice E is an M-space with unit if and only if it is lattice isometric to some $C(\Omega)$ for a (unique up to homeomorphism)Hausdorff compact topological space Ω. In particular, a Banach lattice is an M-space if and only if it is lattice isometric to a closed Riesz subspace of some $C(\Omega)$ space.*

A linear operator $T : E \to F$ between vector lattices E, F is called lattice homomorphism if $T(x \vee y) = Tx \vee Ty$ holds for every $x, y \in E$.A linear operator $T : E \to F$ between vector lattices E, F is said to be positive if $T(E^+) \subseteq F^+$.

Definition 2.3 *Let E, F and G be Archimedean vector lattices. A bilinear map $\Psi : E \times F \to G$ is called positive if $\Psi(x, y) \in G^+$ whenever $x \in E^+, y \in F^+$. A bilinear map $\Psi : E \times F \to G$ is said to be lattice(Riesz) bimorphism if $\Psi(|x|, |y|) = |\Psi(x, y)|$ holds for all $x \in E, y \in F$.*

Given any Archimedean vector lattices E and F , it is constructed an Archimedean vector lattice $E\widetilde{\bigotimes}F$ and a map $\otimes : E \times F \to E\widetilde{\bigotimes}F$ satisfying the following properties:

1. $\bigotimes$ is a Riesz bimorphism and represents $E \bigotimes F$ as a linear subspace of $E\widetilde{\bigotimes}F$.

2. If G is any Archimedean vector lattice, then there is a 1-1 correspondence between Riesz bimorphisms $\Psi : E \times F \to G$ and lattice homomorphism $\tau : E\widetilde{\bigotimes}F \to G$ given by $\Psi = \tau \bigotimes$.

3. $E \bigotimes F$ is dense in $E\widetilde{\bigotimes}F$ in the sense that for any $u \in E\widetilde{\bigotimes}F$ there exist $x_0 \in E^+$, $y_0 \in F^+$ such that for every $\delta > 0$ there is a $v \in E\widetilde{\bigotimes}F$ with $|u - v| \leq \delta x_0 \bigotimes y_0$.

4. If $u \in E\widetilde{\bigotimes}F$, then there exist $x_0 \in E^+$ and $y_0 \in F^+$ such that $|u| \leq x_0 \bigotimes y_0$.

5. $E \bigotimes F$ is order dense in $E\widetilde{\bigotimes}F$ in the sense that whenever $u > 0$ in $E\widetilde{\bigotimes}F$ there exist $x > 0$ in E, $y > 0$ in F such that $0 < x \bigotimes y \leq u$.

6. If G is any Archimedean vector lattice and $\Phi : E \times F \to G$ is a Riesz bimorphism such that $\Phi(x, y) > 0$ whenever $x > 0$ in E and $y > 0$ in F, then $E\widetilde{\bigotimes}F$ may be identified with the Riesz subspace of G generated by $\Phi[E \times F]$.

7. If G is a uniformly complete Archimedean vector lattice, then there is a 1-1 correspondence between positive bilinear maps $\Phi : E \times F \to G$ and increasing linear maps $\tau : E\widetilde{\bigotimes}F \to G$ given by $\Phi = \tau \bigotimes$.

Theorem 2.4 *Riesz(Fremlin) tensor product of two order ideals generated by single elements in a uniformly complete vector lattice is an order ideal.*

Proof. Let E be a uniformly complete vector lattice and $x, y \in E$. Suppose that an order ideal generated by x is U_x and an order ideal generated by y is U_y. These ideals are given by

$U_x = \{z : |z| \leq \lambda|x| \quad for \quad some \quad \lambda > 0\}$ and

$U_y = \{z : |z| \leq \lambda|y| \quad for \quad some \quad \lambda > 0\}$.

Order unit norm on U_x is given by $\|z\| = inf\{\lambda > 0 : |z| \leq \lambda|x|\}$. U_x is algebraically and order isomorphic to an AM-space with unit. Every AM-space with unit is a $C(K)$ space for some compact Hausdorff space K with unit. Similarly, U_y is also algebraically and order isomorphic to an AM-space with unit. It is known that Fremlin tensor product of two AM-space is an AM-space, [7]. Hence, Riesz(Fremlin) tensor product of two order ideals generated by single elements is an order ideal.

For general case, we need the Riesz decomposition property.Then, we can show that Riesz(Fremlin) tensor product of two order ideals is an order ideal.

Theorem 2.5 [1, 10]*(Riesz Decomposition Property)*

If $|x| \leq |y_1 + y_2 + ... + y_n|$ holds in a vector lattice, then there exist $x_1, x_2, ..., x_n$ satisfying $x = x_1 + ... + x_n$ and $|x_i| \leq |y_i|$ for each $i = 1, 2, ..., n$. Moreover, if x is positive, then the x_i can also be chosen to be positive.

Definition 2.6 [6.9] *Let $E_1, E_2, ..., E_n$ and F be vector lattices. A multilinear mapping*

$$T : E_1 \times E_2 \times ... \times E_n \to F$$

is called positive if $T(x_1, x_2, ..., x_n) \geq 0$ for all $x_i \in E_i$ for $i = 1, ..., n$.

Positive multilinear mapping T is denoted by $T \geq 0$.

We say that a multilinear mapping T is regular if T can be written as a difference of two positive multilinear mappings.

Definition 2.7 [6, 9] *Let $E_1, E_2, ..., E_n$ and F be vector lattices. A multilinear mapping*

$$T : E_1 \times E_2 \times ... \times E_n \to F$$

is called order bounded if $T(B_1 \times B_2 \times ... \times B_n)$ is an order bounded set in F for all order bounded sets B_i in E_i for $i = 1, ..., n$.

Definition 2.8 [6, 9] *Let $E_1, E_2, ..., E_n$ and F be vector lattices. A multilinear mapping*

$$T : E_1 \times E_2 \times ... \times E_n \to F$$

is called lattice multimorphism or lattice n-morphism if the linear operator $T^{(j)} : x \to T(x_1, ..., x_{j-1}, x, x_{j+1}, ..., x_n)$ is a lattice homomorphism for any choice of $1 \leq j \leq n$ and $x_k \in E_k^+$, $j \neq k \leq n$.

Definition 2.9 [6, 9]*Let $E_1, E_2, ..., E_n$ and F be vector lattices. A multilinear mapping*

$$T : E_1 \times E_2 \times ... \times E_n \to F$$

is positive if

$$|T(x_1, x_2, ..., x_n)| \leq T(|x_1|, ..., |x_n|)$$

for all $x_i \in E_i$ for $i = 1, 2, ..., n$.

Definition 2.10 [6, 9] *Let $E_1, E_2, ..., E_n$ and F be vector lattices. A multilinear mapping*

$$T : E_1 \times E_2 \times ... \times E_n \to F$$

is said to be a lattice multimorphism if

$$|T(x_1, x_2, ..., x_n)| = T(|x_1|, ..., |x_n|)$$

for all $x_i \in E_i$ for $i = 1, 2, ..., n$.

We refer to [9] for the n-tensor product of Archimedean vector lattices.

3 Main Results

Theorem 3.1 *Let I and J be order ideals in a Riesz space E. Then, the Riesz tensor product $I\widetilde{\bigotimes}J$ of I and J is an order ideal.*

Proof. Case i: Consider the elementary tensors: Assume that

$|u \otimes v| \leq |x \otimes y|$ and $x \otimes y \in I\widetilde{\bigotimes}J$. Let P_1 be a projection. $P_1 : I \times J \to I$ is a lattice bimorphism. Then, there is a lattice homomorphism $\tau_1 : I\widetilde{\bigotimes}J \to I$ such that $P_1 = \tau_1 \bigotimes$. Since a lattice homomorphism is positive, it follows that $\tau_1|u \otimes v| \leq \tau_1|x \otimes y|$ and by the definition of lattice homomorphism we get $|\tau_1(u \otimes v)| \leq |\tau_1(x \otimes y)|$. Hence, we have $|P_1(u, v)| \leq |P_1(x, y)|$ and from here, we get $|u| \leq |x|$. I is an order ideal and $x \in I$, so $u \in I$ by solidness.

Similarly, $|v| \leq |y|$ and $y \in J$ and by solidness of J , we get $v \in J$. Therefore, $u \bigotimes v \in I\widetilde{\bigotimes}J$.

General case ii: For this let $|u \otimes v| \leq |\Sigma_{i=1}^n x_i \otimes y_i|$ and for $x_i \in I, y_i \in J$. Let $P_1 : I \times J \to I$ be projection defined by $P_1(x, y) = x$ and $P_2 : I \times J \to J$ be projection defined by $P_2(x, y) = y$. Projections P_1 and P_2 are lattice bimorphisms. Let τ_1 and τ_2 be corresponding lattice homomorphisms. So, $\tau_1|u \otimes v| \leq \tau_1|\Sigma_{i=1}^n x_i \otimes y_i|$ and $|\tau_1(u \otimes v)| \leq |\Sigma_{i=1}^n \tau_1(x_i \otimes y_i)|$. from here, we get

$|P_1(u, v)| \leq |\sum_{i=1}^n P_1(x_i, y_i|$. So, $|u| \leq |\sum_{i=1}^n x_i|$. By the Riesz Decomposition Property, there exist $u_1, ..., u_n$ satisfying $u = u_1 + ... + u_n$ and $|u_i| \leq |x_i|$ and $x_i \in I$ for $i = 1, ..., n$. By the solidness of I , we get $u_i \in I$ and so $u \in I$. Similarly , $v \in J$. Therefore, $u \otimes v \in I\widetilde{\bigotimes}J$. It is obviously that $I\widetilde{\bigotimes}J$ is subspace. Hence, it is an order ideal.

Theorem 3.2 *Let $I_1, I_2, ..., I_n$ be order ideals in an Archimedean vector lattice E. Then, the Riesz tensor product $I_1 \widetilde{\otimes} I_2 ... \widetilde{\otimes} I_n$ of $I_1, ..., I_n$ is an order ideal.*

Proof: It is a general case of preceding theorem. By the same method, it is proved.

References

[1] C.D. Aliprantis, O. Burkinshaw, *Positive Operators,* Academic Press, London, 1985. https://doi.org/10.1007/978-1-4020-5008-4

[2] M.A. Ben Amor, The Riesz tensor product of d-algebras, *Quaestiones Math.*, (2020), 1-10. https://doi.org/10.2989/16073606.2020.1812756

[3] Y. Azouzi, M.A. Ben Amor, J. Jaber, The tensor product of f-algebras, *Quaestiones Math.,* **41** (3) (2018), 359-369. https://doi.org/10.2989/16073606.2017.1382018

[4] S.J. Bernau, C.B. Huijsmans, Almost f-algebras and d-algebras, *Math. Proc. Cam. Phil. Soc.,* **107** (1990), 287-308. https://doi.org/10.1017/s0305004100068560

[5] G.J.H.M. Buskes, A.W. Wicksted, Tensor product of f-algebras, *Mediter. J. Math.,* (2017), 60-70. https://doi.org/10.1007/s00009-017-0841-x

[6] D.H. Fremlin, Tensor product of Archimedean vector lattices, *Amer. J. Math.,* **94** (1972), 777-798. https://doi.org/10.2307/2373758

[7] D.H. Fremlin, Tensor products of Banach lattices, *Math. Ann.,* **21** (1974), 87-108. https://doi.org/10.1007/bf01344164

[8] J.J. Grobler, C.C.A. Labuschagne, G. Buskes, The tensor product of Archimedean ordered vector spaces, *Math. Proc. Camb. Philos. Soc.,* **104** (1988), 331-345. https://doi.org/10.1017/s0305004100065506

[9] Z. Kusraeva, Sums of disjointness preserving multilinear operators, *Positivity,* **25** (2021), 669-678. https://doi.org/10.1007/s11117-020-00781-7

[10] P. Meyer-Nieberg, *Banach Lattices,* Springer-Verlag, Berlin, 1991. https://doi.org/10.1007/978-3-642-76724-1

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