International Mathematical Forum, Vol. 16, 2021, no. 4, 165 - 170
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https://doi.org/10.12988/imf.2021.912249

Characterization of Automorphisms of Taft Algebras

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Abstract

We shall apply the results that Radford has achieved in order to characterize the automorphisms of the biproducts to Taft algebras and compute the concrete automorphisms of Taft algebras.

Mathematics Subject Classification: 16T05; 16W30

Keywords: Hopf algebra, Biproduct, Automorphism

1 Introduction

Biproducts have a well-established position in the theory of Hopf algebras over a field k. They play a central role in the classification of pointed Hopf algebras [1] and arise quite often in the classification of semisimple Hopf algebras. Given a biproduct $B \times H$, its structure is determined by Hopf algebra maps $\pi : B \times H \to H$ and $j : H \to B \times H$ which satisfy $\pi \circ j = id_H$. A notion of endomorphism (resp. automorphism) of the biproduct $B \times H$ is a Hopf algebra endomorphism (resp. automorphism) F of $B \times H$ which satisfies $\pi \circ F = \pi$ and $F \circ j = j$. In [2], The author relaxed the condition $F \circ j = j$ and characterized the automorphisms of $B \times H$ which only satisfies the condition $\pi \circ F = \pi$.

Taft algebra is an important Hopf algebra, and is isomorphic to a Radford biproduct. The paper will focus on characterizing the automorphisms of Taft algebras which satisfying the suitable conditions as what are considered in [2].

The paper is organized as follows. In Section 2, we recall some basic concepts related to Hopf algebras. In Section 3, we shall characterize the automorphisms of Taft algebras and the concrete corresponding bijection is given.

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2 Preliminary Results

Throughout the paper, we work over a fixed field k and freely use the results, notations, and conventions of [3] and [4]. Let C be a coalgebra, the sigma notation

$$\Delta(c) = c_{(1)} \otimes c_{(2)},$$

for all $c \in C$ will be used frequently later.

2.1 Module algebras

A left H-module algebra is a left H-module $(B, \cdot)$, where B is an algebra over k, such that

$$h \cdot 1_B = \varepsilon(h)1,\ h \cdot (bb') = (h_{(1)} \cdot b)(h_{(1)} \cdot b),$$

for all $h \in H$ and $b, b' \in B$.

Assume that $(B, \cdot)$ is a left H-module algebra, the tensor product $B \otimes H$ of vector spaces has an algebra structure, referred to as the smash product, defined by $1_{B\otimes H} = 1_B \otimes 1_H$ and

$$(b \otimes h)(b' \otimes h') = b(h_{(1)} \cdot b') \otimes h_{(2)}h',$$

for all $b, b' \in B$ and $h, h' \in H$. Typical notation for this algebra is $B\sharp H$ and tensors $b \otimes h$ are written $b\sharp h$.

2.2 Comodule coalgebras

A left H-comodule coalgebra is a left H-comodule C with the comodule structure map $\rho : C \to H \otimes C, \rho(c) = c_{[-1]} \otimes c_{[0]}$, where C is a coalgebra over k, such that

$$c_{[-1]}\varepsilon_C(c_{[0]}) = \varepsilon_C(c)1_H,\ c_{(1)[-1]}c_{(2)[-1]} \otimes c_{(1)[0]} \otimes c_{(2)[0]} = c_{[-1]} \otimes c_{[0](1)} \otimes c_{[0](2)},$$

for all $c \in C$.

Assume that (C, ρ) is a left H-comodule coalgebra, then the tensor product $C \otimes H$ of vector spaces has a coalgebra structure, referred to as the smash coproduct, defined by $\varepsilon_{C\otimes H} = \varepsilon_C \otimes \varepsilon_H$ and

$$\Delta(c \otimes h) = c_{(1)} \otimes c_{(2)[-1]}h_{(1)} \otimes c_{(2)[0]} \otimes h_{(2)},$$

for all $c \in C$ and $h \in H$. Typical notation for this coalgebra is $C\natural H$ and tensors $c \otimes h$ are written $c\natural h$.

2.3 Comodule algebras

A left H-comodule algebra is a left H-comodule (B, ρ), where B is an algebra over k, such that

$$\rho(1_B) = 1_H \otimes 1_B,\ \rho(bb') = b_{[-1]}b'_{[-1]} \otimes b_{[0]}b'_{[0]},\ \forall b, b' \in B.$$

2.4 Module coalgebras

A left H-module coalgebra is a left H-module $(C, \cdot)$, where C is a coalgebra over k, such that

$$\varepsilon_C(h \cdot c) = \varepsilon_H(h)\varepsilon_C(c),\ \Delta(h \cdot c) = h_{(1)} \cdot c_{(1)} \otimes h_{(2)} \cdot c_{(2)},$$

for all $h \in H$ and $c \in C$.

2.5 Biproducts

Let B be both a algebra and a coalgebra (not necessarily a bialgebra). Assume that B is both a left H-module algebra and a left H-comodule coalgebra, then we have the smash product $B\sharp H$ and the smash coproduct $B\natural H$. If the vector space $B \otimes H$ is a bialgebra with the smash product algebra structure and the smash coproduct coalgebra structure, then we call that $B \otimes H$ is a biproduct (or Radford biproduct) of B and H and is denoted $B \times H$. Tensors $b \otimes h$ are denoted $b \times h$.

Theorem 2.1 *The following statements are equivalent:*

- *$B \otimes H$ is a bialgebra;*
- 1. *$\varepsilon_B \in \mathrm{Alg}(B, k)$, $\Delta(1_B) = 1_B \otimes 1_B$.*
  2. *$\Delta(bb') = b_{(1)}(b_{(2)[-1]} \cdot b'_{(1)}) \otimes b_{(2)[0]}b'_{(2)}$, for all $b, b' \in B$.*
  3. *$h_{(1)}b_{[-1]} \otimes h_{(2)} \cdot b_{[0]} = (h_{(1)} \cdot b)_{[-1]}h_{(2)} \otimes (h_{(1)} \cdot b)_{[0]}$.*
  4. *B is both a left H-comodule algebra and a left H-module coalgebra.*

Given a Radford biproduct $B \times H$, we have two Hopf algebra maps

$$\pi : B \times H \to H, \pi(b \times h) = \varepsilon_B(b)h,$$

for $b \in B, h \in H$ and

$$j : H \to B \times H, j(h) = 1_B \times h,$$

for $h \in H$ which satisfy $\pi \circ j = id_H$. Let $\mathrm{End}_{Hopf}(B \times H, \pi)$ be the monoid of all Hopf algebra endomorphisms F of $B \times H$ such that $\pi \circ F = \pi$ and let $\mathrm{Aut}_{Hopf}(B \times H, \pi)$ be its set of units. Thus, $\mathrm{Aut}_{Hopf}(B \times H, \pi)$ is the group of Hopf algebra automorphisms F of $B \times H$ such that $\pi \circ F = \pi$ under composition. In [2], Radford characterized $\mathrm{End}_{Hopf}(B \times H, \pi)$ (resp. $\mathrm{Aut}_{Hopf}(B \times H, \pi)$) as follows:

Theorem 2.2 *Let $B\times H$ be a biproduct, let $\pi : B\times H \to H$ be the projection from $B \times H$ onto H, and let $\mathcal{F}_{B,H}$ be the set of pairs $(\mathcal{L}, \mathcal{R})$, where $\mathcal{L} : B \to B$ and $\mathcal{R} : H \to B$ are maps which satisfy the following conditions:*

(C1) $\mathcal{L}$ *is an algebra endomorphism.*

(C2) $\varepsilon_B \circ \mathcal{L} = \varepsilon_B$,

(C3) $\Delta(\mathcal{L}(b)) = \mathcal{L}(b_{(1)})\mathcal{R}(b_{(2)[-1]}) \otimes \mathcal{L}(b_{(2)[0]}), \forall b \in B.$

(C4) $\rho(\mathcal{L}(b)) = b_{[-1]} \otimes \mathcal{L}(b_{[0]}), \forall b \in B.$

(C4) $\mathcal{L}(h_{(1)} \cdot b)\mathcal{R}(h_{(2)}) = \mathcal{R}(h_{(1)})(h_{(2)} \cdot \mathcal{L}(b)), \forall b \in B$ *and* $h \in H$.

(C5) $\mathcal{R}(hh') = \mathcal{R}(h_{(1)})(h_{(2)} \cdot \mathcal{R}(h')), \forall h, h' \in H.$

(C6) $\mathcal{R}(1_H) = 1_B$.

(C7) $\mathcal{R}$ *is a coalgebra map.*

(C7) $\rho(\mathcal{R}(h)) = h_{(1)}S(h_{(3)}) \otimes \mathcal{R}(h_{(2)}), \forall h \in H.$

Then

1. *The function $\mathcal{F}_{B,H} \to \mathrm{End}_{Hopf}(B \times H, \pi)$, described by $(\mathcal{L}, \mathcal{R}) \mapsto F$, where*
$$F(b \times h) = \mathcal{L}(b)\mathcal{R}(h_{(1)}) \times h_{(2)},$$
for all $b \in B$ and $h \in H$, is a bijection.

2. *Suppose $(\mathcal{L}, \mathcal{R}) \in \mathcal{F}_{B,H}$. Then, $F \in \mathrm{Aut}_{Hopf}(B \times H, \pi)$ if and only if $\mathcal{L}$ is a bijection.*

3 Automorphisms of Taft algebras

Let $n \geq 2$ be a natural number and q an n-th primitive root of unity. The Taft algebra is
$$T_q = k\langle g, x | g^n = 1, x^n = 0, gx = qxy\rangle.$$
The structure of a Hopf algebra on T_q is such that g is group-like, x is $(1, g)$-primitive, that is, $\Delta(x) = x \otimes 1 + g \otimes x$ with $S(x) = -g^{-1}x$. When $n = 2$ note

that we recover Sweedler's Hopf algebra H_4. The Taft algebra is isomorphic to a Radford biproduct

$$T_q \cong k[x]/(x^n)\sharp k\mathbb{Z}_n$$

(send $G \mapsto 1 \otimes g$ and $X \mapsto x \otimes 1$), where $g \cdot x = qx$ and $\rho(x) = g \otimes x$.

For $k[x]/(x^n)\sharp k\mathbb{Z}_n$, we can describe the projection $\pi : k[x]/(x^n)\sharp k\mathbb{Z}_n \to k\mathbb{Z}_n$ as follows:

$$\pi(1 \otimes g^j) = g^j,\ \pi(x^i \otimes g^j) = 0, 1 \le i \le n-1, 0 \le j \le n-1.$$

Now, we shall characterize the automorphisms of the Radford biproduct $k[x]/(x^n)\sharp k\mathbb{Z}_n$.

Theorem 3.1 $\mathcal{R} : k\mathbb{Z}_n \to k[x]/(x^n)$ *which satisfy the conditions of Lemma 3 in [2] is* $\eta \circ \varepsilon$.

Proof Observe that $\{1, g, g^2, \cdots, g^{n-1}\}$ is a basis of $k\mathbb{Z}_n$. For $g^i (1 \le i \le n-1)$, we write $\mathcal{R}(g^i) = a_{i0}1 + a_{i1}x + \cdots + a_{i,n-1}x^{n-1}$. Using (d) of Lemma 3, we have

$$\begin{aligned}\rho(\mathcal{R}(g^i)) &= \rho(a_{i0}1 + a_{i1}x + \cdots + a_{i,n-1}x^{n-1})\\ &= a_{i0}1 \otimes 1 + a_{i1}g \otimes x + \cdots + a_{i,n-1}g^{n-1} \otimes x^{n-1}\end{aligned}$$

and

$$\begin{aligned}g^i S(g^i) \otimes \mathcal{R}(g^i) &= 1 \otimes (a_{i0}1 + a_{i1}x + \cdots + a_{i,n-1}x^{n-1})\\ &= a_{i0}1 \otimes 1 + a_{i1}1 \otimes x + \cdots + a_{i,n-1}1 \otimes x^{n-1}).\end{aligned}$$

Thus it follows that $a_{i1} = \cdots = a_{i,n-1} = 0 (1 \le i \le n-1)$. For any $1 \le i, j \le n-1$, using (a) of Lemma 3, we have that $\mathcal{R}(g^{i+j}) = a_{i+j,0}1$ and

$$\mathcal{R}(g^i)(g^i \cdot \mathcal{R}(g^j)) = a_{i0}1(g^i \cdot a_{j0}1) = a_{i0}a_{j0}.$$

Thus it follows that $a_{i+j,0} = a_{i0}a_{j0}$. Using (c) of Lemma 3, for any g^i, it follows that $a_{i0}^2 = a_{i0}$. By $a_{i+j,0} = a_{i0}a_{j0}$, we can gain $a_{10} = a_{20} = \cdots = a_{n-1,0}$. Furthermore, the desired $\mathcal{R}$ is $\mathcal{R}(g^i) = 1, (0 \le i \le n-1)$, i.e., for all $h \in k\mathbb{Z}_n$, $\mathcal{R}(h) = \varepsilon(h)1$.

From the above theorem and then using Proposition 1 in [2], we can obtain $\mathcal{L} \in \mathrm{Aut}_{\mathcal{YD}\text{-Hopf}}\,(k[x]/(x^n))$. Next, we shall characterize all the $\mathcal{L}$.

Theorem 3.2 $\mathrm{Aut}_{\mathcal{YD}\text{-Hopf}}(k[x]/(x^n))$ *and* $k^\times$ *are isomorphic groups. Precisely, the map*

$$k^\times \to \mathrm{Aut}_{\mathcal{YD}\text{-Hopf}}(k[x]/(x^n)), t \mapsto \mathcal{L}_t,$$

is a group isomorphism, where $\mathcal{L}_t$ *is given by* $\mathcal{L}_t(x^i) = t^i x^i$.

Proof Observe that $\{1, x, x^2, \cdots, x^{n-1}\}$ is a basis of $k[x]/(x^n)$. For $x^i(1 \leq i \leq n-1)$, we write $\mathcal{L}(x^i) = b_{i0}1 + b_{i1}x + \cdots + b_{i,n-1}x^{n-1}$. Using (d) of Lemma 2 in [2], we have

$$\begin{aligned}\rho(\mathcal{L}(x^i)) &= \rho(b_{i0}1 + b_{i1}x + \cdots + b_{i,n-1}x^{n-1}) \\ &= b_{i0}1 \otimes 1 + b_{i1}g \otimes x + \cdots + b_{i,n-1}g^{n-1} \otimes x^{n-1}\end{aligned}$$

and

$$\begin{aligned}g^i \otimes \mathcal{L}(x^i) &= g^i \otimes (b_{i0}1 + b_{i1}x + \cdots + b_{i,n-1}x^{n-1}) \\ &= b_{i0}g^i \otimes 1 + b_{i1}g^i \otimes x + \cdots + b_{i,n-1}g^i \otimes x^{n-1}.\end{aligned}$$

Thus it follows that $b_{ij} = 0, i \neq j$, and $\mathcal{L}(x^i) = b_{ii}x^i$. Since $\mathcal{L}$ is an algebra endomorphism, we can get $b_{i+j,i+j} = b_{ii}b_{jj}$ and $b_{00} = 1$. If b_{ii} $(0 \leq i \leq n-1)$ are subject to the above relationships, the conditions (b), (c) and (e) of Lemma 2 in [2] are naturally satisfied. By $b_{i+j,i+j} = b_{ii}b_{jj}$, it follows that $b_{ii} = (b_{11})^i(1 \leq i \leq n-1)$. For an element $t \in k^\times$, the corresponding element $\mathcal{L}_t(x^i) = t^i x^i$.

Acknowledgements. This work is supported by the National College Students Science and Technology Innovation Project (No. S202010446001).

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Received: June 29, 2021; Published: September 30, 2021