Convergent Infinite Sums and Products
Involving Fibonacci Numbers
Mohammad K. Azarian
Department of Mathematics
University of Evansville
1800 Lincoln Avenue, Evansville, IN 47722, USA

This article is distributed under the Creative Commons by-nc-nd Attribution License.
Copyright © 2021 Hikari Ltd.

Abstract

In this paper we study convergent infinite sums and products involving Fibonacci numbers and identities. We use modifications of some known Fibonacci identities, including Cassini’s, Catalan’s, Lucas’, and Vajda’s identities. Also, we utilize known results from analysis, including properties of convergent infinite series.

Mathematics Subject Classification: 11B39, 05A10, 40A05, 40A20

Keywords: Fibonacci numbers, Fibonacci identities, Fibonacci sequence, golden ratio, Cassini’s identity, Catalan’s identity, Lucas’ identity, Vajda’s identity, Euler number

1 Introduction

The most well-known linear homogeneous recurrence relation of order two with constant coefficients which produces the widely-used Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, ..., is

\[ F_{n+2} = F_{n+1} + F_n, \quad \text{where} \quad F_0 = 0, \ F_1 = F_2 = 1, \ \text{and} \ n \geq 0. \]
There are various approaches to finding $\lim_{n \to \infty} \frac{F_{n+1}}{F_n}$, $n \geq 1$. In [11], with a few straight-forward calculations, we obtained the following closed-form expression for the $n$th element of the Fibonacci sequence

$$F_n = \frac{1}{\sqrt{5}}[(\frac{1 + \sqrt{5}}{2})^n - (\frac{1 - \sqrt{5}}{2})^n], n \geq 0.$$ From this explicit value of $F_n$, we get

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \frac{1 + \sqrt{5}}{2} = \phi,$$
and

$$\lim_{n \to \infty} \frac{F_{n+m}}{F_n} = \lim_{n \to \infty} \frac{F_n}{F_{n-m}} = (\lim_{n \to \infty} \frac{F_{n+1}}{F_n})^m = \phi^m,$$

where $m$ is a positive integer. Throughout the paper, $\phi$ will represent the golden ratio.

In [2-4], we used the formula

$$F_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}, \quad n \geq 0$$

to write any nonzero Fibonacci number as a finite binomial sum, and we used this formula to present some Fibonacci and Lucas identities as binomial sums as well. Also, $F_{n+1}$ can be written as an infinite binomial sum

$$F_{n+1} = \sum_{i=0}^{\infty} \binom{n-i}{i}, \quad n \geq 0,$$

where only a finite number of binomial coefficients are different than zero. Moreover, in [5], we found some relationships between Fibonacci numbers and the Euler number $e$. Additionally, in [9], we used Fibonacci numbers to deal with the staircase problem.

Our goal in this paper is to present some convergent infinite series as well as some convergent infinite products involving Fibonacci numbers and identities. We use modifications of known Fibonacci identities, including Cassini’s, Catalan’s, Lucas’, and Vajda’s identities. Also, we utilize known results from analysis, including properties of convergent infinite series.

To make this paper somewhat self-contained, we list known results that will be used in the paper in the following two theorems. In Theorem 1.1, we list six known results from analysis, and in Theorem 1.2, we state five known Fibonacci identities as follows.
Theorem 1.1. (i) [12, p. 241]. If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \sqrt{a_n/n^p}$ converges for $p > \frac{1}{2}$.

(ii) [12, p. 91]. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of positive terms, then $\sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}$ converges also.

(iii) [12, p. 83]. Let $\{a_n\}$ be a sequence of positive numbers. Then the infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ and the infinite sum $\sum_{n=1}^{\infty} a_n$ converge or diverge together.

(iv) [12, p. 98]. The series $\sum_{n=1}^{\infty} \left(\frac{1}{n^p} + \frac{1}{n^p} - \frac{1}{n^p} + \frac{1}{n^p} - \ldots - \frac{1}{n^p} + \frac{1}{n^p}\right)$ converges if and only if $p > 2$.

(v) [12, p. 129]. Let $a > 1$. Then
$$\sum_{n=0}^{\infty} \frac{2^n}{1 + a^{2^n}} = \frac{1}{a - 1}.$$

(vi) [1, pp. 5 and 53]. Let $p_1, p_2, \ldots, p_n, \ldots$ be positive real numbers. If the series $\sum_{n=1}^{\infty} \frac{1}{p_n}$ is convergent, then the series
$$\sum_{n=1}^{\infty} \frac{n^2 p_n}{(p_1 + p_2 + \ldots + p_n)^2}$$
is also convergent.

Theorem 1.2. Let $k$, $n$, and $r$ be positive integers.

(i) Catalan’s identity ($n \geq r$):
$$F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r}F_r^2.$$

(ii) Cassini’s identity (a special case of Catalan’s identity, when $r = 1$):
$$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}.$$

(iii) Lucas’ identities:

(a) $\sum_{i=1}^{n+1} F_i^2 = F_{n+1}F_{n+2}$. 
(b) $F_{n+1}^2 + F_n^2 = F_{2n+1}$.

(iv) Vajda’s identity:

$$F_n F_{n+k+r} - F_{n+k} F_{n+r} = -(F_{n+k} F_{n+r} - F_n F_{n+k+r}) = (-1)^{n+1} F_k F_r.$$ 

(v) The following two identities are well-known (for example, see [19, pp. 183-84]):

(a) $$1 + \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{F_{i+1} F_i} = \phi;$$

(b) $$\prod_{i=1}^{\infty} \left[ 1 + \frac{(-1)^{i+1}}{F_{i+1}^2} \right] = \phi.$$ 

2 Results

In this section we state and prove eight theorems. For proofs we utilize Theorems 1.1 and 1.2. We refer the reader to [15] and [19] for known results regarding Fibonacci numbers and identities.

**Theorem 2.1.** Let $F_k$ ($k \geq 4$) be any Fibonacci number. Then the infinite series

$$\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{n} (\frac{2i-1}{2i})^{\frac{F_k}{2}}}{n^{\frac{1}{F_k}}}$$

converges.

**Proof.** By Theorem 1.1 (iv), $\sum_{n=1}^{\infty} \prod_{i=1}^{n} \left( \frac{2i-1}{2i} \right)^{F_k}$ converges. Now, since $\frac{1}{F_k} > \frac{1}{2}$, by Theorem 1.1 (i), the given infinite series converges, provided we let $a_n = (\prod_{i=1}^{n} \frac{2i-1}{2i})^{\frac{1}{F_k}}$ and $p = F_k^{\frac{1}{F_k}}$. \hfill \Box

**Remark 2.2.** Theorem 2.1 can also be proven as follows: If $k \geq 3$, then $F_k \geq 2$, and $F_k^{\frac{1}{F_k}} = p > 1$. Now, since $\prod_{i=1}^{n} \frac{2i-1}{2i} < 1$, we deduce that

$$\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{n} (\frac{2i-1}{2i})^{\frac{F_k}{2}}}{n^{p}} < \sum_{n=1}^{\infty} \frac{1}{n^{p}}.$$
Therefore, the given infinite series converges, by the basic Comparison Test.

We invite the reader to provide alternative proofs for theorems which we present in this article.

**Theorem 2.3.** Let \( n \geq 2 \). Then

\[
\lim_{n \to \infty} \frac{F_{2n+1} + 2F_{n+1}F_n - \sum_{i=1}^{n+1} F_i^2}{F_{n+1}^2 - F_n^2} = \varphi.
\]

**Proof.** By Theorem 1.2 (iii)(b),

\[
F_{2n+1} + 2F_{n+1}F_n = F_{n+1}^2 + F_n^2 + 2F_{n+1}F_n = (F_{n+1} + F_n)^2 = F_{n+2}^2.
\]

Now, by Theorem 1.2 (iii)(a) and the fact that

\[
F_{n+1}^2 - F_n^2 = F_{n+2}F_{n-1},
\]

we have

\[
\lim_{n \to \infty} \frac{F_{2n+1} + 2F_{n+1}F_n - \sum_{i=1}^{n+1} F_i^2}{F_{n+1}^2 - F_n^2} = \lim_{n \to \infty} \frac{F_{n+2} - F_{n+1}F_{n+2}}{F_{n+2}F_{n-1}} = \lim_{n \to \infty} \frac{F_{n+2} - F_{n+1}}{F_{n-1}} = \varphi. \quad \Box
\]

**Theorem 2.4.** Let \( k \) and \( r \) be positive integers. Then the following double infinite series both converge to the reciprocal of the golden ratio:

\[
(i) \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} \frac{F_{i+k+1}F_{i+r+1} - F_{i+1}F_{i+k+r+1}}{F_kF_rF_{i+1}F_i} n;
\]

\[
(ii) \sum_{k=0}^{\infty} [(F_{2n}^2 - F_{2n+1}F_{2n-1})\sum_{i=1}^{\infty} (F_{i+2}^2 - F_{i+3}F_{i+1})(\sum_{j=1}^{i} F_j^2)^{-1}]^k.
\]

**Proof (i).** By Theorem 1.2 (iv),

\[
\frac{F_{i+k+1}F_{i+r+1} - F_{i+1}F_{i+k+r+1}}{F_kF_rF_{i+1}F_i} = -\frac{(-1)^{i+1}F_kF_r}{F_kF_rF_{i+1}F_i} = -\frac{(-1)^{i+1}}{F_{i+1}F_i}.
\]
Now, by Theorem 1.2 (v)(a), we have
\[
\lim_{i \to \infty} \sum_{i=1}^{\infty} F_{i+k+1}F_{i+r+1} - F_{i+1}F_{i+k+r+1} = \sum_{i=1}^{\infty} (-1)^{i+1} = 1 - \varphi.
\]
Hence,
\[
\sum_{n=0}^{\infty} (1 - \varphi)^n = \frac{1}{\varphi}.
\]

**Proof (ii).** From Theorem 1.2 (ii) and 1.2 (iii)(a), we have
\[
\sum_{n=0}^{\infty} \sum_{i=1}^{\infty} \left( \frac{(n^2 + F_{n+k})^2 + (n^2 + F_{n+r})^2}{n^2 + F_{n+k}} \right) = \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} \left( \frac{(-1)^{i+1}}{F_{i+1}F_i} \right)^k.
\]
Hence, by Theorem 1.2 (v)(a)
\[
\sum_{n=0}^{\infty} \left( \frac{(-1)^{i+1}}{F_{i+1}F_i} \right)^k = \sum_{n=0}^{\infty} (1 - \varphi)^k = \frac{1}{\varphi}.
\]

**Theorem 2.5.** Let \( k \) and \( r \) be integers such that \( r > k \geq 0 \). Then the following infinite series and the infinite product both converge:

(i) \( \sum_{n=1}^{\infty} \frac{\sqrt{(n^2 + F_{n+k})^2 + (n^2 + F_{n+r})^2}}{(n^2 + F_{n+k})(n^2 + F_{n+r})} \);

(ii) \( \prod_{n=1}^{\infty} \frac{n^4 + (n^2 + 1)F_{n+r} + (n^2 - 1)F_{n+k + F_{n+k}F_{n+r}}}{(n^2 + F_{n+k})(n^2 + F_{n+r})} \).

**Proof.** Since \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is a convergent \( p \)-series, by the basic Comparison Test both \( \sum_{n=1}^{\infty} \frac{1}{n^2 + F_{n+k}} \) and \( \sum_{n=1}^{\infty} \frac{1}{n^2 + F_{n+r}} \) converge. Now, from this observation and Theorem 1.1 (ii), the infinite series (i) converges. Also, since both \( \sum_{n=1}^{\infty} \frac{1}{n^2 + F_{n+k}} \) and \( \sum_{n=1}^{\infty} \frac{1}{n^2 + F_{n+r}} \) are convergent, we deduce that their difference
\[
\sum_{n=1}^{\infty} \frac{1}{n^2 + F_{n+k}} - \sum_{n=1}^{\infty} \frac{1}{n^2 + F_{n+r}} = \sum_{n=1}^{\infty} \frac{F_{n+r} - F_{n+k}}{(n^2 + F_{n+k})(n^2 + F_{n+r})}
\]
is also convergent. Therefore, by Theorem 1.1 (iii)
\[ \prod_{n=1}^{\infty} \left[ 1 + \frac{F_{n+r} - F_{n+k}}{(n^2 + F_{n+k})(n^2 + F_{n+r})} \right] = \prod_{n=1}^{\infty} \frac{n^4 + (n^2 + 1)F_{n+r} + (n^2 - 1)F_{n+k} + F_{n+k}F_{n+r}}{(n^2 + F_{n+k})(n^2 + F_{n+r})} \]
converges, provided we let \( a_n = \frac{F_{n+r} - F_{n+k}}{(n^2 + F_{n+k})(n^2 + F_{n+r})} \). □

**Theorem 2.6.** Let \( r \geq 0 \). The following double infinite series is convergent:
\[ \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} n^2 \frac{\sum_{i=1}^{\lfloor r/2 \rfloor} (r-j)}{F_n F_{2k} \sum_{j=0}^{\lfloor r/2 \rfloor} (r-j)} \text{ for } k \leq \frac{1}{2} \]

**Proof.** The given expression can be rewritten as
\[ \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} n^2 \frac{F_{r+1}}{F_n F_{2k}} \left( \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \cdots + \frac{1}{F_n} \right) \]
\[ = \sum_{k=1}^{\infty} F_{r+1} \left( \sum_{n=1}^{\infty} \frac{n^2 \left( \frac{2^n}{n F_n} \right)}{F_n F_{2k}} \right) \]
Now, since \( \sum_{n=1}^{\infty} \frac{n F_n}{2^n} \) is a convergent series, by Theorem 1.1 (vi), the infinite series
\[ \sum_{n=1}^{\infty} \frac{n^2 \left( \frac{2^n}{n F_n} \right)}{\left( \frac{2}{F_1} + \frac{4}{F_2} + \frac{8}{F_3} + \cdots + \frac{2^n}{n F_n} \right)^2} \]
is convergent, provided we let \( p_n = \frac{2^n}{n F_n} \). Finally, since \( \sum_{k=1}^{\infty} \frac{1}{F_{2k}^2} \) is convergent, the given double infinite series is convergent as well. □

**Theorem 2.7.** The following infinite series is convergent:
\[ \sum_{n=1}^{\infty} \frac{n^2 [(F_2 + F_4 + \cdots + F_{2n})^2 F_n + (1 - F_{n+2})^2 F_{2n}]}{[(1 - F_{n+2})(F_2 + F_4 + \cdots + F_{2n})]^2} \]

**Proof.** First we note that the given expression can be rewritten as
\[ \sum_{n=1}^{\infty} \frac{n^2 F_n}{(1 - F_{n+2})^2} + \sum_{n=1}^{\infty} \frac{n^2 F_{2n}}{(F_2 + F_4 + \cdots + F_{2n})^2} \]
\[ = \sum_{n=1}^{\infty} \frac{n^2 F_n}{(F_1 + F_2 + \cdots + F_n)^2} + \sum_{n=1}^{\infty} \frac{n^2 F_{2n}}{(F_2 + F_4 + \cdots + F_{2n})^2} \]
Now, from the fact that \( \sum_{n=1}^{\infty} \frac{1}{F_n} \) is a convergent series and Theorem 1.1 (vi), the first infinite series is convergent, provided we let \( p_n = F_n \). Also, from the fact that \( \sum_{n=1}^{\infty} \frac{1}{F_{2n}} \) is a convergent series and Theorem 1.1 (vi), the second infinite series is convergent, provided we let \( p_n = F_{2n} \). Therefore, the sum of these two infinite series converges as well. □

Theorem 2.8. The double infinite sum

\[
\sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \left[ \frac{1}{1 + (2 + F_{2i+1})^{2n}} + \frac{1}{1 + (1 + F_{2i})^{2n}} \right] 2^{n+1}
\]
converges to 7.

Proof. The given infinite sum can be rewritten as

\[
2 \sum_{i=0}^{\infty} \left[ \sum_{n=0}^{\infty} \frac{2^n}{1 + (2 + F_{2i+1})^{2n}} + \sum_{n=0}^{\infty} \frac{2^n}{1 + (1 + F_{2i})^{2n}} \right].
\]

Next, if we first let \( a = 2 + F_{2i+1} \), and then let \( a = 1 + F_{2i} \) in Theorem 1.1 (v), the above double sum can be rewritten as

\[
2 \sum_{i=0}^{\infty} \left( \frac{1}{1 + F_{2i+1}} + \frac{1}{F_{2i}} \right) = 2 \sum_{i=0}^{\infty} \frac{1}{1 + F_{2i+1}} + 2 \sum_{i=0}^{\infty} \frac{1}{F_{2i}}.
\]

Now, it is well-known that

\[
\sum_{i=0}^{\infty} \frac{1}{1 + F_{2i+1}} = \frac{\sqrt{5}}{2} \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{1}{F_{2i}} = \frac{7 - \sqrt{5}}{2}.
\]

Therefore,

\[
2 \sum_{i=0}^{\infty} \frac{1}{1 + F_{2i+1}} + 2 \sum_{i=0}^{\infty} \frac{1}{F_{2i}} = 7. \quad \Box
\]

Theorem 2.9. Let \( k \) and \( r \) be positive integers. Then the following infinite products both converge to the golden ratio:

(i) \[
\prod_{i=1}^{\infty} \frac{F_{2i+1}^2 - F_{i+2}^2 - F_i^2}{2(\sum_{j=1}^{i} F_j^2 - F_{i+1}F_{i+2})}
\]

(ii) \[
\prod_{i=1}^{\infty} \frac{F_{2i+1}F_iF_kF_i^2 + F_iF_{i+k+r} - F_{i+k}F_{i+r} - F_{2i+2}F_{2i}F_{i+k+1}}{F_kF_iF_{i+1}^2}.
\]
**Proof of (i).** From Theorem 1.2 (iii)(a), we have

\[
\frac{F_{i+1}^2 - F_{i+2}^2 - F_i^2}{2(\sum_{j=1}^{i} F_j^2 - F_{i+1}F_{i+2})} = \frac{F_{i+1}^2 - F_{i+2}^2 - F_i^2}{-2F_{i+1}^2}
\]

\[
= \frac{-F_{i+1}^2 + F_{i+2}^2 + F_i^2}{2F_{i+1}^2} = \frac{2F_{i+1}^2 + (F_{i+2}^2 - 3F_{i+1}^2 + F_i^2)}{2F_{i+1}^2}
\]

\[
= \frac{F_{i+1}^2 + \frac{1}{2}(F_{i+2}^2 - 3F_{i+1}^2 + F_i^2)}{F_{i+1}^2}
\]

\[
= (1 + \frac{1}{2}(F_{i+2}^2 - 3F_{i+1}^2 + F_i^2)).
\]

Now, from the fact that (for example, see [19, p. 26])

\[
\frac{1}{2}(F_{i+2}^2 - 3F_{i+1}^2 + F_i^2) = (-1)^{i+3} = (-1)^{i+1},
\]

we have

\[
(1 + \frac{1}{2}(F_{i+2}^2 - 3F_{i+1}^2 + F_i^2)) = (1 + \frac{(-1)^{i+1}}{F_{i+1}^2}).
\]

Therefore, by Theorem 1.2 (v)(b), we obtain

\[
\prod_{i=1}^{\infty} \frac{F_{i+1}^2 - F_{i+2}^2 - F_i^2}{2(\sum_{j=1}^{i} F_j^2 - F_{i+1}F_{i+2})} = \prod_{i=1}^{\infty} [1 + \frac{(-1)^{i+1}}{F_{i+1}^2}] = \varphi. \quad \square
\]

**Proof of (ii).** First we note that

\[
\frac{F_{2r+1}^2 F_{i} F_k F_{i+1}^2 + F_i F_{i+k+r} - F_{i+k} F_{i+r} - F_{2r+2} F_{2r} F_k F_{i+1}^2}{F_k F_r F_{i+1}^2}
\]

\[
= \frac{(F_{2r+1}^2 - F_{2r+2} F_{2r}) F_k F_r F_{i+1}^2 + F_i F_{i+k+r} - F_{i+k} F_{i+r}}{F_k F_r F_{i+1}^2}
\]

\[
= F_{2r+1}^2 - F_{2r+2} F_{2r} + \frac{F_i F_{i+k+r} - F_{i+k} F_{i+r}}{F_k F_r F_{i+1}^2}.
\]

Next, by Theorem 1.2 (ii) and 1.2 (iv), we get

\[
F_{2r+1}^2 - F_{2r+2} F_{2r} + \frac{F_i F_{i+k+r} - F_{i+k} F_{i+r}}{F_k F_r F_{i+1}^2}
\]

\[
= 1 + \frac{(-1)^{i+1} F_k F_r}{F_k F_r F_{i+1}^2} = 1 + \frac{(-1)^{i+1}}{F_{i+1}^2}.
\]
Finally, by Theorem 1.2 (v)(b), we obtain
\[
\prod_{i=1}^{\infty} \left[ 1 + \frac{(-1)^{i+1}}{F_{i+1}^2} \right] = \varphi. \quad \square
\]

**Question 2.10.** (i) What are the values of the convergent infinite sums in Theorems 2.1, 2.5(i), 2.6, and 2.7? (ii) What is the value of the convergent infinite product in Theorem 2.5(ii)?

**References**


Convergent infinite sums and products involving Fibonacci numbers


Received: February 9, 2021; Published: February 26, 2021