Derivation of Lobachevsky Integrals in Terms of Special Functions

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Abstract

We consider some integrals first derived by Nikolai Lobachevsky. We evaluate several of these definite integrals of the form

$$\int_0^{\infty} \frac{R(y)}{\cos(b) + \cosh(\alpha y)} dy$$

in terms of a special function, where $R(y)$ is a general complex function and $b$ and $\alpha$ are arbitrary complex numbers.

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1 Introduction

We will derive integrals as indicated in the abstract in terms of special functions. Some special cases of these integrals have been reported in Gradshteyn and Ryzhik [7]. In 1835, Lobachevsky [9] derived hyperbolic integrals of the form

\[ \int_0^\infty \frac{R(y)}{\cos(b) + \cosh(\alpha y)} dy \]

In our case the constants in the formulas are general complex numbers subject to the restrictions given below. The derivations follow the method used by us in [8]. The generalized Cauchy’s integral formula is given by

\[ \frac{y^k}{k!} = \frac{1}{2\pi i} \int_C \frac{e^{wy}}{w^{k+1}} dw. \tag{1} \]

This method involves using a form of equation (1) then multiply both sides by a function, then take a definite integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of equation (1) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

2 Definite integral of the contour integral

We use the method in [8]. We replace the variable \( w \) in the integration over the contour by \( u = m + w \). The cut and contour is in the upper left quadrant of the \( u \)-plane with \( u > -|Re(b)| \). The cut is vertical at \( Im(u) = \infty \) and approaches the origin from the interior of the quadrant. The contour is on opposite sides of the cut going round the origin with zero radius. There is one point where \( u = 0 \) but at this point the series given by (1.232.3) in [7] is infinite which gives the correct value for the cosech. Using a generalization of Cauchy’s integral formula we first replace \( y \) by \( iy + \log(a) \) then multiply both sides by \( e^{imy} \) to get the first equation. Then we replace \( y \) by \(-iy + \log(a) \) then multiply both sides by \( e^{-imy} \) to get the second equation. Then we add these two equations, followed by multiplying both sides by \( \frac{1}{2(\cos(b) + \cosh(\alpha y))} \) to get

\[ \frac{e^{-imy}(-iy + \log(a))^k + e^{imy}(iy + \log(a))^k}{2\Gamma(k + 1)(\cos(b) + \cosh(\alpha y))} = \frac{1}{2\pi i} \int_C a^w w^{-k-1} \frac{\cos(uy)}{\cos(b) + \cosh(\alpha y)} du \tag{2} \]

the logarithmic function is defined in equation (4.1.2) in [3]. The variables \( \alpha, a, b, k \) and \( m \) are general complex numbers. We then take the definite integral over \( y \in [0, \infty) \) of both sides to get
Derivation of Lobachevsky integrals in terms of special functions

\[\frac{1}{2\Gamma(k+1)} \int_0^\infty e^{-imy} \frac{(-iy + \log(a))^k + e^{imy}(iy + \log(a))^k}{(\cos(b) + \cosh(\alpha y))} \, dy\]

\[= \frac{1}{2\pi i} \int_0^\infty \int_C a^w w^{-k-1} \frac{\cos(uy)}{\cos(b) + \cosh(\alpha y)} \, du \, dy\]

\[= \frac{1}{2\pi i} \int_C \left( \int_0^\infty \frac{\cos(uy)}{\cos(b) + \cosh(\alpha y)} \, dy \right) \frac{a^w \, dw}{w^{k+1}}\]

\[= \frac{1}{2i\alpha \sin(b)} \int_C a^w w^{-k-1} \csc \left( \frac{\pi u}{\alpha} \right) \sinh \left( \frac{bu}{\alpha} \right) \, du\]

from equation (1.7.7.7) in [2] and the integral is valid for \(\alpha, m, a, k\) and \(b\) complex and \(\text{Re}(b) < \pi\).

3 Infinite sum of the contour integral

Again, using the method in [8], and equation (1) we will form two equations by first replacing \(y\) with \(b + y\) then multiplying both sides by \(e^{bm/\alpha}\). Then setting \(b = -b\) to get the second equation followed by adding these two equations. Using the result of this new equation, we replace \(y\) with \(\pi(2p + 1)/\alpha + \log(a)\) and multiply both sides by

\[- \frac{\pi}{\alpha \sin(b)} e^{m\pi(2p+1)/\alpha}\]

to yield

\[\frac{\pi \csc(b) \left( \frac{1}{\alpha} \right)^{k+1} e^{m(-b+2\pi p + \pi)\alpha}}{\Gamma(k+1)} \left( (\alpha \log(a) - b + 2\pi p + \pi)^k - e^{2bm\alpha}(\alpha \log(a) + b + 2\pi p + \pi)^k \right)\]

\[= - \frac{1}{i\alpha} \int_C a^w \csc(b) w^{-k-1} e^{\frac{\pi(2p+1)u}{\alpha}} \sinh \left( \frac{bu}{\alpha} \right) \, du\]

followed by taking the infinite sum of both sides of equation (4) with respect to \(p\) over \([0, \infty)\) to get
\[
\frac{2^k \pi^{k+1}}{\Gamma(k+1)} \csc(b) \left( \frac{1}{\alpha} \right)^{k+1} e^{\frac{(\pi-b)m}{\alpha}} \left( \Phi \left( e^{\frac{2m\pi}{\alpha}}, -k, \frac{-b + \alpha \log(a) + \pi}{2\pi} \right) \right)
\]
\[
- e^{\frac{2bm}{\alpha}} \Phi \left( e^{\frac{2m\pi}{\alpha}}, -k, \frac{b + \alpha \log(a) + \pi}{2\pi} \right)
\]
\[
= - \frac{1}{4ib \sin(b)} \sum_{p=0}^{\infty} \int_{C} a^w w^{-k-1} e^{\frac{\pi(2p+1)u}{\alpha}} \sinh \left( \frac{bu}{\alpha} \right) du
\]
\[
= - \frac{1}{4ib \sin(b)} \int_{C} \sum_{p=0}^{\infty} a^w w^{-k-1} e^{\frac{\pi(2p+1)u}{\alpha}} \sinh \left( \frac{bu}{\alpha} \right) du
\]
\[
= \frac{1}{2i\alpha \sin(b)} \int_{C} a^w w^{-k-1} \text{csch} \left( \frac{\pi u}{\alpha} \right) \sinh \left( \frac{bu}{\alpha} \right) du
\]

from (1.232.3) in [7] and \( \text{Re}(u) < 0 \) for the convergence of the sum and if the \( \text{Re}(k) < 0 \) then the argument of the sum over \( p \) cannot be zero for some value of \( p \). We use (9.550) and (9.556) in [7] where \( \Phi(z, s, v) \) is the Lerch function which is a generalization of the Hurwitz Zeta and polylogarithm functions.

The Lerch function has a series representation given by
\[
\Phi(z, s, v) = \sum_{n=0}^{\infty} (v + n)^{-s} z^n
\]

where \( |z| < 1, v \neq 0, -1, .. \) and is continued analytically by its integral representation given by
\[
\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} e^{-vt} \frac{1 - z e^{-t}}{1 - z} dt = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} e^{-(v-1)t} dt
\]

where \( \text{Re}(v) > 0 \), or \( |z| \leq 1, z \neq 1, \text{Re}(s) > 0 \), or \( z = 1, \text{Re}(s) > 1 \).

Since the right hand-side of equation (3) is equal to the right-hand side of (5), we can equate the left hand-sides and simplify to get
\[
\int_{0}^{\infty} \frac{(e^{imy} \log(a) + iy)^k + e^{-imy} \log(a) - iy)^k}{\cos(b) + \cosh(\alpha y)} dy
\]
\[
= (2\pi)^{k+1} \csc(b) \left( \frac{1}{\alpha} \right)^{k+1} e^{\frac{(\pi-b)m}{\alpha}} \left( \Phi \left( e^{\frac{2m\pi}{\alpha}}, -k, \frac{-b + \alpha \log(a) + \pi}{2\pi} \right) \right)
\]
\[
- e^{\frac{2bm}{\alpha}} \Phi \left( e^{\frac{2m\pi}{\alpha}}, -k, \frac{b + \alpha \log(a) + \pi}{2\pi} \right)
\]

where \( \text{Re}(b) > 0 \) and \( k, m, \alpha \) and \( a \) are general complex numbers.
4 Special Cases

Using equation (8) we formulate two equations by setting $a = 1$, taking $m$ and replacing it by $-m$ and adding, where

$$I = 2^{k-1} \pi^{k+1} \csc(b) \left( \frac{1}{\alpha} \right)^{k+1} e^{-\frac{i(k+\pi)m}{\alpha}} \sec \left( \frac{\pi k}{2} \right)$$

them to get

$$\int_0^\infty \frac{y^k \cosh(my)}{\cos(b) + \cosh(\alpha y)} dy = I \left( e^{\frac{2ibm}{\alpha}} \Phi \left( e^{-\frac{2im\pi}{\alpha}}, -k, \frac{\pi - b}{2\pi} \right) \right.
- \Phi \left( e^{-\frac{2im\pi}{\alpha}}, -k, \frac{b + \pi}{2\pi} \right) )
+ I \left( e^{\frac{2im\pi}{\alpha}} \Phi \left( e^{\frac{2im\pi}{\alpha}}, -k, \frac{\pi - b}{2\pi} \right)
- e^{\frac{2ibm}{\alpha}} \Phi \left( e^{\frac{2im\pi}{\alpha}}, -k, \frac{b + \pi}{2\pi} \right) \right) \left(9\right)$$

Note that $\zeta(s,v) = \Phi(1,s,v)$ from equations (9.521.1) and (9.550) in [7]. We also note the Lerch formula for the alternating Hurwitz zeta function $\Phi(-1,s,v) = \zeta^*(s,v)$ given by

$$\Phi(-1,s,v) = \sum_{n=0}^{\infty} (v + n)^{-s} (-1)^n$$

$$= 2^s \left( \zeta(s,\frac{v}{2}) - \zeta(s,\frac{1+v}{2}) \right) \left(10\right)$$

from [1], p.25, where $Re(s) < 1$ and $0 < v \leq 1$.

4.1 Derivation of entry 3.527.4 in [7]

Using equation (9) we first set $m = 0$ then take the partial derivative with respect to $b$ then set $b = \pi/2$ to get

$$\int_0^\infty y^k \text{sech}^2(\alpha y) dy = -2^{k-1} k \pi^k \left( \frac{1}{\alpha} \right)^{k+1} \left( \zeta \left( 1 - k, \frac{1}{4} \right) \right.
+ \zeta \left( 1 - k, \frac{3}{4} \right) \sec \left( \frac{\pi k}{2} \right) \left(11\right)$$

where $\Phi(1,s,a) = \zeta(s,a)$. 
Next we take the limit of equation (11) as $k \to 1$ and using L'Hôpital's rule to get

$$\lim_{k \to 1} \int_0^\infty y^k \text{sech}^2(\alpha y) dy = \lim_{k \to 1} \left( -2^{k-1} k \pi \left( \frac{1}{\alpha} \right)^{k+1} \left( \zeta \left( 1 - k, \frac{1}{4} \right) + \zeta \left( 1 - k, \frac{3}{4} \right) \sec \left( \frac{\pi k}{2} \right) \right) \right)$$

$$= \frac{1}{\left( -\frac{\pi}{2} \right)} \left( -\pi \left( -\zeta^\prime \left( 0, \frac{1}{4} \right) - \zeta^\prime \left( 0, \frac{3}{4} \right) \right) \right) \frac{\log(2)}{\alpha^2}$$

from page 179 in [4].

4.2 Derivation of entry 3.531.2 in [7]

Using equation (9) and setting $m = 0$, $\alpha = 2$ and $b = 2t$ and taking the limit as $k \to 1$ and using L'Hôpital's rule to get

$$\int_0^\infty \frac{y \cos(2t) + \cosh(2y)}{\csc(2t) + \cosh(2y)} dy = \frac{4 \pi \csc(2t)}{\alpha^2} \left( \zeta^\prime \left( -1, \frac{1}{2} - \frac{t}{\pi} \right) - \zeta^\prime \left( -1, \frac{t}{\pi} + \frac{1}{2} \right) \right)$$

(13)

where the variable $t$ is continued analytically because of the $\zeta$ function.

4.3 Derivation of entry 3.533.1 in [7]

Using equation (9) and setting $m = 1$, $b = 2t - \pi$ and $\alpha = 2$ we get

$$\int_0^\infty \frac{y^k \cosh(y)}{\cosh(2y) - \cos(2t)} dy = 2^{k-2} \pi^{k+1} e^{-it} \left( 1 + e^{2it} \sec \left( \frac{\pi k}{2} \csc(2t) \left( \zeta \left( -k, \frac{\pi - t}{2\pi} \right) \right) + \zeta \left( -k, t \frac{2\pi}{2\pi} \right) \right) - 2^{k-2} \pi^{k+1} e^{-it} \left( 1 + e^{2it} \sec \left( \frac{\pi k}{2} \csc(2t) \left( \zeta \left( -k, \frac{t + \pi}{2\pi} \right) \right) + \zeta \left( -k, 1 - \frac{t}{2\pi} \right) \right) \right)$$

(14)
Next we take the limit of equation (14) as as $k \to 1$ and using L'Hôpital's rule to get

$$
\int_0^\infty \frac{y \cosh(y)}{\cosh(2y) - \cos(2t)} \, dy = \pi \csc(t) \left( \zeta\left(-1, \frac{\pi - t}{2\pi}\right) + \zeta\left(-1, \frac{t}{2\pi}\right) \right) \\
- \pi \csc(t) \left( \zeta\left(-1, \frac{t + \pi}{2\pi}\right) + \zeta\left(-1, 1 - \frac{t}{2\pi}\right) \right)
$$

(15)

where the variable $t$ is continued analytically because of the $\zeta$ function.

4.4 Derivation of entry 4.371.1 in [7]

Using equation (9) setting $m = 0$ then taking the first partial derivative with respect to $k$ then setting $k = 0$, $b = \pi/2$ and $\alpha = 1$ we get

$$
\int_0^\infty \log(x) \text{sech}(x) \, dx = \frac{1}{2} \pi \left( 2\zeta\left(0, \frac{3}{4}\right) - 2\zeta\left(0, \frac{1}{4}\right) \right) + \frac{1}{2} \pi \log(2) + \frac{1}{2} \pi \log(\pi)
$$

$$
= \pi \log \left( \frac{\sqrt{2\pi} \Gamma\left(-\frac{1}{4}\right)}{3 \Gamma\left(-\frac{3}{4}\right)} \right)
$$

(16)

from equation (1.13) in [6].

4.5 Derivation of entry 4.371.2 in [7]

Using equation (9) setting $m = 0$ then taking the first partial derivative with respect to $k$ then setting $k = 0$, $b = t$ and $\alpha = 1$ we get

$$
\int_0^\infty \frac{\log(x)}{\cos(t) + \cosh(x)} \, dx = \csc(t) \left( t \log(2\pi) + \pi \log \left( \frac{\Gamma\left(\frac{t + \pi}{2\pi}\right)}{\Gamma\left(\frac{t - \pi}{2\pi}\right)} \right) \right)
$$

(17)

from equation (1.13) in [6].

4.6 Derivation of entry 4.371.3 in [7]

Using equation (9) setting $m = 0$ then taking the first partial derivative with respect to $b$ then setting $b = \pi/2$ and $\alpha = 1$. Next we take the first partial derivative with respect to $k$ we get
\[
\int_0^\infty x^k \log(x) \sech^2(x) \, dx \\
= 4^{-k} \left( \zeta(k) \Gamma(k) \left( 2 \left( 2^k + k \left( \frac{2^k - 2}{2} \log(\pi) + \log(4) - 2 \right) \right) \right) \\
+ 4^{-k} \left( \pi \left( 2^k - 2 \right) k \tan \left( \frac{\pi k}{2} \right) \right) \left( 2^k - 2 \right) k(2\pi)^k \sec \left( \frac{\pi k}{2} \right) \zeta'(1 - k) \right) 
\]

(18)

Next we use L'Hôpital's rule to get

\[
\int_0^\infty \log(x) \sech^2(x) \, dx = \log \left( \frac{\pi}{4} \right) - \gamma 
\]

(19)

where \( \gamma \) is Euler's constant.

### 4.7 Derivation of entry 4.373.1 in [7]

Using equation (9) setting \( m = 0 \) and \( b = \pi/2 \) we get

\[
\int_0^\infty \sech(\alpha x) \left( (\log(a) - ix)^k + (\log(a) + ix)^k \right) \, dx \\
= (2\pi)^{k+1} \left( \frac{1}{\alpha} \right)^{k+1} \left( \zeta(\frac{-k}{2\pi}) \frac{\alpha \log(a) + \frac{\pi}{2}}{2\pi} \right) - \zeta(\frac{-k}{2\pi}) \frac{\alpha \log(a) + \frac{3\pi}{2}}{2\pi} 
\]

(20)

Next we take the first partial derivative with respect to \( k \) and set \( k = 0 \), \( \alpha = b \) and \( a = e^a \) simplify the to get

\[
\int_0^\infty \log \left( a^2 + x^2 \right) \sech(bx) \, dx \\
= \frac{\pi}{b} \log \left( \frac{(2ab - \pi) \Gamma \left( \frac{ab}{2\pi} - \frac{1}{4} \right)}{(2ab - 3\pi) \Gamma \left( \frac{ab}{2\pi} - \frac{3}{4} \right)} \right) + \log \left( \frac{2\pi}{b} \right) 
\]

(21)

from equation (1.13) in [6].

### 4.8 Derivation of entry 4.373.2 in [7]

Using equation (21) setting \( a = 1 \) and \( b = \pi/2 \) then using L'Hôpital's rule as \( b \to \pi/2 \) to get

\[
\int_0^\infty \log \left( x^2 + 1 \right) \sech \left( \frac{\pi x}{2} \right) \, dx = \log \left( \frac{16}{\pi^2} \right) 
\]

(22)
4.9 Derivation of entry 3.527.6 in [7]

Using equation (9) replacing $m$ by $a$, $\alpha$ by $a$ and setting $b = \pi/2$ simplifying the Lerch function to the Hurwitz zeta function we get

\[
\int_{0}^{\infty} x^{k+1} \tanh(ax) \text{sech}(ax)dx = 2^{k}(k + 1)\pi^{k+1} \left( \frac{1}{a} \right)^{k+2} \left( \zeta \left( -k, \frac{1}{4} \right) - \zeta \left( -k, \frac{3}{4} \right) \right) \sec \left( \frac{\pi k}{2} \right) \tag{23}
\]

where $k$ is continued analytically by the Hurwitz zeta function.

4.10 Derivation of entry 3.527.7 in [7]

Using equation (23) then applying L'Hôpital’s rule as $k \to 0$ we get

\[
\int_{0}^{\infty} x \tanh(ax) \text{sech}(ax)dx = \frac{\pi}{2a^2} \tag{24}
\]

4.11 Derivation of entry 3.527.14 in [7]

Using equation (23) and setting $k = a = 1$ we get

\[
\int_{0}^{\infty} x^2 \tanh(x) \text{sech}(x)dx = 4K \tag{25}
\]

where $K$ is Catalan’s constant. The constant is named in honor of E. C. Catalan (1814-1894), who first gave an equivalent series and expressions in terms of integrals.

5 Conclusion

In this paper, we have presented our method for deriving some interesting definite integrals by Nikolai Lobachevsky using contour integration. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

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