Mannheim B-Curves in Weyl Space

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Abstract

We have defined Mannheim B-curves and Mannheim B-pair in three dimensional Weyl space $W_3$. Under the condition that the pair $(C, C^*)$ is a Mannheim B-pair in $W_3$, we have given some theorems such as the relation between Bishop vector fields of $C$ and $C^*$; the relation between Frenet vector fields of $C$ and $C^*$; the relation between Bishop curvatures of $C$ and $C^*$; the relation between Frenet curvatures of $C$ and $C^*$.

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1 Introduction

Mannheim curves have been presented by Mannheim in 1878 and by Blum [3] in 1966. After that, Mannheim partner curves have been studied in Euclidean 3-space by Liu and Wang [8] in 2008 and by Orbay and Kasap [12] in 2009. Besides, Mannheim partner curves in dual space have been examined by Özkaldı et al. [15] in 2009. Generalized Mannheim curves have been given in Minkowski space-time $E^4_1$ by Akyiğit et al. [1] in 2011 and in Euclidean 4-space by Matsuda and Yorozu [10] in 2009. Later, Mannheim offsets of ruled surfaces have been defined by Orbay et al. [13] in 2009 and dual Mannheim partner curves have been expressed by Güngör and Tosun [4] in 2010. Furthermore, the quaternionic Mannheim curves in Euclidean 4-space, weakened Mannheim curves and

Recently, Mannheim curves have been defined according to different frame such as Mannheim partner D-curves (Önder and Kızıltuğ, 2012) [14] and Mannheim B-curves (Masal and Azak, 2017) [9].

2 Preliminaries

Let $C$ be a curve in three dimensional Weyl space $W_3$. Let $\{v^1, v^2, v^3, \kappa_1, \kappa_2\}$ $\{v^1, n^1, t^1, k_1, k_2\}$ be the Frenet and Bishop apparatus [2] of $C$, respectively. Then, Frenet and Bishop formulas of $C$ are expressed in the following form:

\[
\begin{align*}
\nabla_k v^1 &= \kappa_1 v^2 \\
\nabla_k v^2 &= -\kappa_1 v^1 + \kappa_2 v^3 \\
\nabla_k v^3 &= -\kappa_2 v^2 
\end{align*}
\]  

(1)

and

\[
\begin{align*}
\nabla_k v^1 &= k_1 n^1 + k_2 n^2 \\
\nabla_k n^1 &= -k_1 v^1 \\
\nabla_k n^2 &= -k_2 v^1. 
\end{align*}
\]  

(2)

Besides, the relation between Frenet and Bishop vector fields [7] and the curvatures of Frenet and Bishop [7] can be written as follows:

\[
\begin{bmatrix}
v^1 \\
v^2 \\
v^3
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
v^1 \\
n^1 \\
n^2
\end{bmatrix}
\]  

(3)

and $\kappa_1 = \frac{2}{3}, \kappa_2 = -\frac{2}{3}, \kappa_2 = v^k \nabla_k \theta, k_1 = \kappa_1 \cos \theta, k_2 = \kappa_1 \sin \theta, k_1^2 + k_2^2 = \kappa_1^2, k_1 = \frac{2}{3} \cos \theta - \frac{2}{3} \sin \theta, k_2 = \frac{2}{3} \sin \theta + \frac{2}{3} \cos \theta$, where $r_s = T_k v^k$ and $r_s T_k$ are called geodesic curvature and Chebyshev curvature of the first kind of the net $(v, n, t)$ [16].

3 Mannheim B-Curves in Weyl Space

Let $C$ and $C^*$ be curves in three dimensional Weyl space $W_3$. Let us denote arc-length of $C$ and $C^*$ by $s$ and $s^*$, respectively. Then, we can express the
curves $C$ and $C^*$ in the form $C : x^i = x^i(s)$ and $C^* : \dot{x}^i = \dot{x}^i(s^*)(i = 1, 2, 3)$, respectively. Let us denote Bishop apparatus of $C$ and $C^*$ by \{${v^i_1, n^i_1, n^i_2, k_1, k_2}$\} \{${\star v^i_1, \star n^i_1, \star n^i_2, \star k_1, \star k_2}$\}, respectively.

**Definition 3.1.** If the Bishop vector field $n^i_1$ coincides with the Bishop vector field $\star n^i_2$ at the corresponding points of curves $C$ and $C^*$, then the curve $C$ is called a Mannheim partner B-curve of $C^*$ and $(C, C^*)$ is also called Mannheim B-pair.

This can be formalized as

\[ C(s) = C^*(s^*) + \lambda(s^*)\hat{n}^i_2(s^*) \]  
(4)

or

\[ x^i(s) = \dot{x}^i(s^*) + \lambda(s^*)\dot{\hat{n}}^i_2(s^*) \]  
(5)

where $\lambda$ is a function of $s^*$, see Figure 1.

![Figure 1: Mannheim B-Curves](image)

Let us obtain the relation between the Bishop vector fields of $C$ and $C^*$ such that the pair $(C, C^*)$ is a Mannheim B-pair:

Since $v^i_1$ is orthogonal to $n^i_1$ and $n^i_1 = \star n^i_2$, $\dot{v}^i$ is orthogonal to $\dot{\hat{n}}^i_2$. Therefore, $v^i_1$ can be written as

\[ v^i_1 = \gamma_1v^i_1 + \gamma_2\hat{n}^i_2 \]  
(6)

or

\[ v^i_1 = \cos \alpha \hat{v}^i_1 + \sin \alpha \hat{n}^i_1 \]  
(7)
where $\alpha = \angle (v_1^i, v_1^i)$, $\gamma_1 = g_{ij} v_1^i v_1^j = \cos \alpha$ and $\gamma_2 = g_{ij} n_1^i n_1^j = \cos (\frac{\pi}{2} - \alpha) = \sin \alpha$.

Since $n_2^i$ is orthogonal to $n_1^i$ and $n_1^i = \star v_1^i$, $n_1^i$ is orthogonal to $\star n_2^i$. Then, $n_2^i$ can be written as

$$n_2^i = \eta_1 v_1^i + \eta_2 n_1^i$$

(8)

or

$$n_2^i = - \sin \alpha v_1^i + \cos \alpha n_1^i$$

(9)

where $\eta_1 = g_{ij} n_1^i v_1^j = \cos (\frac{\pi}{2} + \alpha) = - \sin \alpha$ and $\eta_2 = g_{ij} n_2^i n_1^j = \cos \alpha$.

By means of (7) and (9), we can express this relation among Bishop vector fields in the following matrix form:

$$\begin{pmatrix} v_1^i \\ n_1^i \\ n_2^i \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ 0 & 0 & 1 \\ -\sin \alpha & \cos \alpha & 0 \end{pmatrix} \begin{pmatrix} \star v_1^i \\ \star n_1^i \\ \star n_2^i \end{pmatrix}.$$  

(10)

**Theorem 3.2.** If the pair $(C, C^*)$ is a Mannheim B-pair then $\lambda$ is a non-zero constant.

**Proof.** Let $(C, C^*)$ be a Mannheim B-pair. Then (5) is satisfied.

If we take the prolonged covariant derivative of (5) in the direction of $v_1^k$, we get

$$v_1^k \nabla_k x^i = v_1^k \nabla_k x^i + (v_1^k \nabla_k \lambda) n_2^i + \lambda v_1^k \nabla_k n_2^i = (1 - \lambda k_2) v_1^i + (v_1^k \nabla_k \lambda) n_2^i.$$  

(11)

Let us denote left hand side of (11) by

$$v_1^k \nabla_k x^i = (v_1^k \nabla_k x^i). A.$$  

(12)

By using (12) in (11), we have

$$v_1^i . A = (1 - \lambda k_2) v_1^i + (v_1^k \nabla_k \lambda) n_2^i.$$  

(13)

Multiplying (13) by $g_{ij} v_1^j$ and summing on $i$ and $j$, we obtain

$$A = (1 - \lambda k_2) \cos \alpha$$  

(14)
where $g_{ij} \mathring{v}_j^i = g_{ij} \mathring{n}_j^i = 0$.

From (13) and (14), we have

$$
u^i(1 - \lambda k_2) \cos \alpha = (1 - \lambda k_2)\nu^i + (\mathring{v}_1^k \nabla_k \lambda)\mathring{n}_2^i. \quad (15)$$

Multiplying (15) by $g_{ij} \mathring{n}_2^j$ and summing on $i$ and $j$, we obtain

$$\mathring{v}_1^k \nabla_k \lambda = 0 \quad (16)$$

where $g_{ij} \mathring{v}_j^i = g_{ij} \mathring{n}_j^i = 0$, $g_{ij} \mathring{v}_1^i \mathring{n}_2^j = 0$ and $g_{ij} \mathring{n}_1^i \mathring{n}_2^j = 1$.

Thus, $\lambda$ is a non-zero constant. \hfill \Box

**Theorem 3.3.** If $(C, C^*)$ is a Mannheim $B$-pair in $W_3$, then there are the following relations between Bishop vector fields of $C$ and $C^*$:

$$
u^i = \varepsilon \mathring{v}_1^i, \quad n^i_1 = \varepsilon \mathring{n}_2^i, \quad n_2^i = \varepsilon \mathring{n}_2^i$$

where $\varepsilon = \pm 1$ for $\alpha = 0$ and $\alpha = \pi$.

**Proof.** Let $(C, C^*)$ be a Mannheim B-pair in $W_3$. Then $\lambda$ is a non-zero constant. So, we get from (13)

$$\nu^i A = (1 - \lambda k_2)\mathring{v}_1^i. \quad (17)$$

Multiplying (17) by $g_{ij} \mathring{v}_1^i$ and summing on $i$ and $j$, we have

$$A \cos \alpha = 1 - \lambda k_2 \quad (18)$$

where $g_{ij} \mathring{v}_1^i \mathring{v}_1^j = \cos \alpha$ and $g_{ij} \mathring{v}_2^i \mathring{v}_2^j = 1$.

Using (14) in (18), we get $\cos^2 \alpha = 1$. That is, $\cos \alpha = 1$ for $\alpha = 0$ and $\cos \alpha = -1$ for $\alpha = \pi$. So, $\sin \alpha = 0$ is obtained.

If obtained results are used in equation (10), we have

$$\nu^i = \varepsilon \mathring{v}_1^i, \quad n^i_1 = \varepsilon \mathring{n}_2^i, \quad n_2^i = \varepsilon \mathring{n}_2^i$$

where $\cos \alpha = \varepsilon = \pm 1$ for $\alpha = 0$ and $\alpha = \pi$. \hfill \Box

**Theorem 3.4.** Let $(C, C^*)$ be a Mannheim $B$-pair in $W_3$. Let us denote their Frenet apparatus by $\{v_1^i, v_2^i, v_3^i, \kappa_1, \kappa_2\}$ and $\{\mathring{v}_1^i, \mathring{v}_2^i, \mathring{v}_3^i, \mathring{\kappa}_1, \mathring{\kappa}_2\}$ in $W_3$, respectively. The relations between Frenet vector fields of $C$ and $C^*$ are given by

$$
\mathring{v}_1^i = \varepsilon v_1^i \\
\mathring{v}_2^i = \sin(\theta^* - \varepsilon \theta)v_2^i - \varepsilon \cos(\theta^* - \varepsilon \theta)v_3^i \\
\mathring{v}_3^i = \cos(\theta^* - \varepsilon \theta)v_2^i - \varepsilon \sin(\theta^* - \varepsilon \theta)v_3^i
$$

where $\varepsilon = \pm 1$ for $\alpha = 0$ and $\alpha = \pi$. 
Proof. Let \((C, C^*)\) be a Mannheim B-pair in \(W_3\). Let us express (3) for Frenet and Bishop apparatus of \(C^*\):

\[
\begin{pmatrix}
\ast^i v_1 \\
\ast^i v_2 \\
\ast^i v_3
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta^* & \sin \theta^* \\
0 & -\sin \theta^* & \cos \theta^*
\end{pmatrix}
\begin{pmatrix}
\ast^i v_1 \\
\ast^i n_1 \\
\ast^i n_2
\end{pmatrix}
\]

(19)

where \(\theta^* = \angle(\ast^i v_2, \ast^i n_1)\).

Using (3), (4) and Theorem 3.3, we get

\[
\begin{pmatrix}
\ast^i v_1 \\
\ast^i v_2 \\
\ast^i v_3
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\cos \alpha & 0 & 0 \\
0 & 0 & -\cos \alpha \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\ast^i v_1 \\
\ast^i v_2 \\
\ast^i v_3
\end{pmatrix}
\]

(20)

or

\[
\ast^i v_1 = \cos \alpha \ast^i v_1
\]
\[
\ast^i v_2 = \{\sin \theta^* \cos \theta - \sin \theta \cos \alpha \cos \theta^*\} \ast^i v_2 + \{-\sin \theta^* \sin \theta - \cos \theta \cos \alpha \cos \theta^*\} \ast^i v_3
\]
\[
\ast^i v_3 = \{\cos \theta^* \cos \theta + \sin \theta \sin \theta^* \cos \alpha\} \ast^i v_2 + \{-\cos \theta^* \sin \theta + \cos \theta \sin \theta^* \cos \alpha\} \ast^i v_3
\]

(21)

or

\[
\ast^i v_1 = \varepsilon \ast^i v_1
\]
\[
\ast^i v_2 = \sin(\theta^* - \varepsilon \theta) \ast^i v_2 - \varepsilon \cos(\theta^* - \varepsilon \theta) \ast^i v_3
\]
\[
\ast^i v_3 = \cos(\theta^* - \varepsilon \theta) \ast^i v_2 + \varepsilon \sin(\theta^* - \varepsilon \theta) \ast^i v_3
\]

(22)

where \(\varepsilon = \cos \alpha; \varepsilon = 1\) for \(\alpha = 0, \varepsilon = -1\) for \(\alpha = \pi\).

\[\square\]

**Theorem 3.5.** Let \((C, C^*)\) be a Mannheim B-pair in \(W_3\). Then the relation between the second Bishop curvature \(k_2^*\) of the curve \(C^*\) and the first Bishop curvature \(k_1\) of the curve \(C\) is as follows:

\[
k_2^* = \frac{k_1}{1 + \lambda k_1}.
\]
Proof. Let \((C, C^\star)\) be a Mannheim B-pair in \(W_3\). Then \(v_1^i A = (1 - \lambda k_2) v_1^i\) is satisfied.

Taking prolonged covariant derivative of this equality in the direction of \(v_1^i\), we have

\[
(v^k k_1^i v_1^i) A^2 + v_1^i (v^k k_1^i v_1^i) = -\lambda (v^k k_1^i v_1^i) + (1 - \lambda k_2)(v^k k_1^i v_1^i),
\]

or

\[
(k_1 n^i_1 + k_2 n^i_2) A^2 + v_1^i (v^k k_1^i v_1^i) = -\lambda (v^k k_1^i v_1^i) + (1 - \lambda k_2)(k_1 n^i_1 + (1 - \lambda k_2) k_2 n^i_2).
\]

Multiplying (24) by \(g_{ij} n^i_1 n^j_1\) and summing on \(i\) and \(j\), we get

\[
k_1 A^2 = (1 - \lambda k_2) k_2
\]

where \(g_{ij} n^i_1 n^j_1 = 1, g_{ij} n^i_1 n^j_2 = 0, g_{ij} n^i_2 n^j_1 = 0, g_{ij} n^i_2 n^j_2 = 0,\) and \(g_{ij} n^i_2 n^j_2 = 1,\)

Using (18) in (24), we get

\[
k_1 = \frac{k_2}{1 - \lambda k_2} \quad \text{or} \quad k_2 = \frac{k_1}{1 + \lambda k_1}.
\]

The proof is completed. 

\[\square\]

Remark 3.6. Using [7] in (26), \(k_2\) can also be expressed in the following form:

\[
k_2 = \frac{\frac{2}{1} \cos \theta - \frac{3}{1} \sin \theta}{1 + \lambda (\frac{2}{1} \cos \theta - \frac{3}{1} \sin \theta)}.
\]

Theorem 3.7. Let \((C, C^\star)\) be a Mannheim B-pair in \(W_3\). Then the following equalities are satisfied:

\[
k_1 A = \cos \alpha k_2
\]

\[
k_1 + k_2 A = 0.
\]
Proof. Let \((C, C^\star)\) be a Mannheim B-pair. Then \(v^i = \cos \alpha \v^i_1\) is obtained from (10) and Theorem 3.3.

Taking prolonged covariant derivative of this equality in the direction of \(\v^k_1\), we get

\[
(v^k_1 \nabla^k_1 v^i) A = \cos \alpha \v^k_1 \nabla^k_1 \v^i_1
\]

or

\[
(k_1 n^i_1 + k_2 n^i_2) A = \cos \alpha (k_1 n^i_1 + k_2 n^i_2).
\]

Multiplying (29) by \(g^s_1 n^j_1\) and summing on \(i\) and \(j\), we have

\[
-k_2 A = k_1
\]

or

\[
\v^i_1 = k_1 + k_2 A = 0
\]

where

\[
\v^i_1 = \v^i_2 = \epsilon_{ijk} v^j_1 \cos \alpha v^k_1 = -\cos \alpha \epsilon_{ijk} v^j_1 n^i_1 = -\cos \alpha n^i_2.
\]

Multiplying (29) by \(g^s_1 n^j_2\) and summing on \(i\) and \(j\), we have

\[
k_1 A = \cos \alpha \v^i_2
\]

where \(g^s_1 n^j_2 = g^s_1 n^j_1 = 1, g^s_2 n^j_2 = g^s_2 n^j_1 = 0, g^s i^j_1 n^j_1 = 0 \) and \(g^s i^j_2 n^j_2 = 1\).

Remark 3.8. Using [7], (27) and (30), \(k_1\) can also be expressed in the following form:

\[
k_1 = -\frac{2}{1 \cos \theta + \frac{3}{1 \sin \theta}} \cos \alpha.
\]

Theorem 3.9. Let \((C, C^\star)\) be a Mannheim B-pair in \(W_3\). Then there is a relation between the first Frenet curvatures of \(C\) and \(C^\star\) in the following form:

\[
\v^i_1 = \kappa_1 A.
\]
Proof. Let \((C, C^*)\) be a Mannheim B-pair in \(W_3\). Using (30) and (32), we get
\[
\begin{align*}
\kappa_1^2 + \kappa_2^2 &= (k_1^2 + k_2^2)A^2 \\
\kappa_1^2 &= \kappa_1^2 A^2 \\
\kappa_1 &= \kappa_1 |A|.
\end{align*}
\] (34)

Remark 3.10. Using [7] and (34), we obtain
\[
\frac{2}{\tilde{c}_1^i} = \frac{2}{\tilde{c}_1} |A|
\] (35)
where \(\kappa_1 = \frac{2}{\tilde{c}_1} = \tilde{T}_k v^k_1\) and \(\kappa_1 = \frac{2}{\tilde{c}_1} = \tilde{T}_k v^k\).

Theorem 3.11. Let \((C, C^*)\) be a Mannheim B-pair in \(W_3\). Then
\[
\tilde{c}_1^i = c_1^i (1 - \lambda \kappa_2)
\] (39)
is satisfied where \(\tilde{c}_1^i\) are the geodesic vector fields of the net \((\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)\) occurred by the vector fields of \(C^*\) and \(c_1^i\) are the geodesic vector fields of the net \((v_1, v_2, v_3)\) occurred by the vector fields of \(C\) in \(W_3\) [16].

Proof. Let \((C, C^*)\) be a Mannheim B-pair in \(W_3\).

Taking prolonged covariant derivative of the first equation in (22) in the direction of \(\tilde{v}_1^k\), we have
\[
\tilde{v}_1^k \tilde{\nabla}_k \tilde{v}_1^i = \varepsilon \tilde{v}_1^k \tilde{\nabla}_k \tilde{v}_1^i = \varepsilon (v_1^k \tilde{\nabla}_k v_1^i) A
\] (36)
\[
\tilde{T}_k v^k_1 v^i_1 = \varepsilon \tilde{T}_k v^k_1 v^i_1 A \quad (p = 1, 2, 3)
\] (37)
\[
\tilde{T}_v^i = \varepsilon \tilde{T}_v^i A
\] (38)
\[
\tilde{c}_1^i = \varepsilon c_1^i A.
\] (39)

Using (14) in (39), we get
\[
\frac{2}{\tilde{c}_1^i} = \frac{2}{\tilde{c}_1} (1 - \lambda \kappa_2).
\] (40)

\]
By means of (27) and Theorem 3.11, we can give the following theorem:

**Theorem 3.12.** Let $(C, C^*)$ be a Mannheim $B$-pair in $W_3$. The net $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is a geodesic net in $W_3$ if and only if the net $(v_1, v_2, v_3)$ is a geodesic net in $W_3$.

**References**


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