An Inertial Method for Solving Split Monotone Variational Inclusion Problems in Hilbert Spaces\textsuperscript{1}

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Abstract

In this paper, an inertial iteration is proposed to find a solution of split monotone variational inclusion problem in Hilbert spaces, and the strong convergence theorem of the sequence generated by the algorithm is obtained. Further, some numerical examples are presented to show the efficiency of this algorithm.

Mathematics Subject Classifications: 47H09, 47H10, 47J25

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1 Introduction and preliminaries

Let $C$ and $Q$ be nonempty closed and convex subsets of real Hilbert spaces $H_1$ and $H_2$ respectively, $A : H_1 \to H_2$ be a bounded linear operator. Let $B_1 : H_1 \rightrightarrows H_1$ and $B_2 : H_2 \rightrightarrows H_2$ be two multivalued operators with nonempty values, and let $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be two operators.

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For modeling inverse problems that arise from phase retrievals and medical reconstruction, Censor and Elfving [1] introduced so-called split feasibility problem in 1994, which is to Find a point \( x^* \in C \) such that \( Ax^* \in Q \).

Motivated by this work, in 2011, Moudafi [2] introduced the following split monotone variational inclusion problems (Shortly, SMVIP):

\[
\text{find } x^* \in H_1 \text{ such that } 0 \in f(x^*) + B_1(x^*), \tag{1.1}
\]

and

\[
\text{find } y^* = Ax^* \in H_2 \text{ solves } 0 \in g(y^*) + B_2(y^*), \tag{1.2}
\]

To solve SMVIP, Moudafi proposed the following algorithm: for \( \gamma > 0 \) and \( x_0 \in H_1 \),

\[
x_{k+1} = U(x_k + \gamma A^*(T - I)Ax_k), \quad k \in N, \tag{1.3}
\]

where \( A^* \) is the adjoint operator of \( A \), \( \gamma \in [0, \frac{1}{L}] \) with \( L \) being the spectral radius of operator \( A^*A \), \( T := J_{\lambda}^{B_2}(I - \lambda g) \) and \( U := J_{\lambda}^{B_1}(I - \lambda f) \), \( J_{\lambda}^{B_1} \) and \( J_{\lambda}^{B_2} \) are the resolvent operators of \( B_1 \) and \( B_2 \) respectively. Under suitable conditions, Moudafi proved a weak convergence theorem of this iteration.

Given a nonempty closed convex subset \( C \) of \( H \), the normal cone of \( C \) at \( x \in C \) is given by

\[
N_C(x) := \{ z \in H : \langle z, v - x \rangle \leq 0, \quad \forall v \in C \}. \tag{1.4}
\]

Replacing \( B_1(x) \) and \( B_2(x) \) by \( N_C(x) \) and \( N_Q(x) \) in (1.1)-(1.2) respectively, then SMVIP reduces to split variational inequality problem. Besides, there are many other special cases of SMVIP, such as split minimization problem, split equilibrium problem and split feasibility problem [1,3–5]. These problems have been widely applied in intensity-modulated radiation therapy treatment planning, sensor networks and image reconstruction [6–9].

Recently, for accelerating the convergence rate of iteration algorithm, researchers own increasingly interests in inertial type algorithm because the effect of this method on accelerating the convergence rate is obvious and has been showed in related results, including inertial Mann algorithm, inertial Douglas-Rachford splitting method and inertial extragradient method [11–15]. The inertial algorithm originated from the heavy ball method while Polyak [10] proposed an inertial extrapolation based on a discrete version of the two-order time dynamical system to solve the smooth convex minimization problem. In [16], Alvarez and Attouch proposed a inertial proximal point algorithm for finding a zero of maximal monotone operator, which is as follows: for any \( x_0, x_1 \in H \),

\[
\text{find } x_{n+1} \in H \text{ such that } 0 \in r_nB(x_{n+1}) + x_{n+1} - x_n - \theta_n(x_n - x_{n-1}), \tag{1.5}
\]
which can be also written as:

$$\text{find } x_{n+1} \in H \text{ such that } x_{n+1} = (I + r_n B)^{-1}(x_n + \theta_n(x_n - x_{n-1})), \quad (1.6)$$

where \(\{r_n\}\) and \(\{\theta_n\}\) are two sequences of parameters satisfying some suitable conditions. They proved that the sequence converges weakly to a zero of the maximal monotone operator \(B\). In 2015, Lorenz and Pock [17] extended this method to inertial forward-backward algorithm for solving the zero point problem of sum of two maximal monotone operators:

$$x_{n+1} = (I + r_n B)^{-1}(I + r_n A)(x_n + \theta_n(x_n - x_{n-1})), \quad n \geq 1, \quad (1.7)$$

where \(A\) and \(B\) are two maximal monotone operators. Under suitable conditions, they also proved the sequence \(\{x_n\}\) converges weakly to a solution of this problem. Very recently, Cholamjiak et al [18] proposed a new algorithm combining iteration (1.7) with Halpern iteration method, which is written as:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \beta_n)u + \beta_n(I + r_n B)^{-1}(I + r_n A)w_n, \end{cases} \quad (1.8)$$

where \(u\) is an arbitrarily fixed point in \(H\). They obtained a strong convergence theorem for the sequence generated by (1.8).

Since Moudafi only got weak convergence of the solution of SMVIP in [2], in this paper, we construct an inertial iteration method and obtain a strong convergence theorem for the split monotone variational inclusion problem in Hilbert spaces. Finally, we give numerical examples to illustrate the efficiency of our results.

Let \(H\) be a Hilbert space with inner product \(\langle \cdot, \cdot \rangle\) and norm \(\| \cdot \|\), \(C\) be a nonempty closed convex subset of \(H\). We denote the strong convergence and weak convergence of a sequence \(\{x_n\}\) to a point \(x \in H\) by \(x_n \to x\) and \(x_n \rightharpoonup x\), respectively. For every point \(x \in H\), there exists a unique nearest point of \(C\) denoted by \(P_C x\), such that \(\|x - P_C x\| \leq \|x - y\|\) for all \(y \in C\), and the \(P_C\) is called the metric projection from \(H\) onto \(C\). It is well known that for any \(x \in H\) and \(z \in C, z = P_C x\) if and only if

$$\langle x - z, z - y \rangle \geq 0, \quad \forall y \in C. \quad (1.9)$$

We recall some definitions and lemmas that will be used in our proofs.

**Definition 1.1.** An operator \(T : H \to H\) is said to be nonexpansive iff for all \(x, y \in H\),

$$\|Tx - Ty\| \leq \|x - y\|. \quad (1.10)$$
And an operator $T : H \to H$ is said to be firmly nonexpansive iff for all $x, y \in H$, 
\[ \|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle. \]  
(1.11)

**Definition 1.2.** An operator $T : H \to H$ is said to be averaged iff it can be written as 
\[ T = (I - \alpha)I + \alpha S, \]  
(1.12)
where $\alpha \in [0, 1]$ and $S : H \to H$ is nonexpansive.

**Definition 1.3.** An operator $T : H \to H$ is said to satisfy the demi-closedness principle if for any sequence $\{x_n\}$ that $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$ implies $(I - T)x = 0$.

We note that every averaged operator is also a nonexpansive operator, and every nonexpansive operator satisfies the demi-closedness principle in metric spaces (for more details, see [20]).

**Definition 1.4.** An operator $T : H \to H$ is said to be $\alpha$-inverse iff for any $x, y \in D(A)$, there exists a constant $\alpha > 0$ such that 
\[ \langle Tx - Ty, x - y \rangle \geq \alpha \|Tx - Ty\|^2. \]  
(1.13)

Now let us recall that an operator $B : H \rightharpoonup H$ with domain $D(B) := \{ x \in H : B(x) \neq \emptyset \}$ is said to be monotone iff $\langle x - y, u - v \rangle \geq 0$ for all $u \in Bx$ and $v \in By$. In addition, it is called maximal monotone iff its graph $G(B) := \{(x, u) : x \in D(B), u \in Bx\}$ can not be properly contained by any other monotone operators. The resolvent operator $J_B^\lambda : H \to H$ of the maximal monotone operator $B$ is defined by $J_B^\lambda := (I + \lambda B)^{-1}$ for $\lambda > 0$ and $I$ is an identity operator on $H$. It is well known that $J_B^\lambda$ is single-valued and firmly nonexpansive.

**Lemma 1.5.** [19] Denote the fixed point set of an operator $T : H \to H$ by $\text{Fix}(T)$. If the operators $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then 
\[ \bigcap_{i=1}^N \text{Fix}(T_i) = \text{Fix}(T_1 \cdots T_n). \]  
(1.14)

**Lemma 1.6.** [21] Let $H$ be a real Hilbert space. Then for all $x, y \in H$ and $t \in R$, following equation and inequality hold:

1. $\|x + y\|^2 \leq \|x\|^2 + 2 \langle x + y, y \rangle$;
2. $\|tx + (1 - t)y\|^2 = t\|x\|^2 - t(1 - t)\|x - y\|^2 + (1 - t)\|y\|^2$. 
Lemma 1.7. [2] Let $H_1$ and $H_2$ be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator, and $A^*$ be the adjoint operator of $A$. Let $B_1 : H_1 \rightrightarrows H_1$ and $B_2 : H_2 \rightrightarrows H_2$ be two maximal monotone operator and $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be $\beta_1$ and $\beta_2$ inverse strongly monotone operators, respectively. Set $T := J^{B_2}_\lambda(I - \lambda g)$ and $U := J^{B_1}_\lambda(I - \lambda f)$ with $\lambda \in [0, 2\beta]$ where $\beta := \min\{\beta_1, \beta_2\}$. Then the operators $U$ and $I - \gamma A^*(I - T)A$ are averaged, so as $U(I - \gamma A^*(I - T)A)$.

Lemma 1.8. [22, 23] Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n \to \infty} \gamma_n \leq 0$. Suppose that

$$s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n \gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$ Then $\lim_{n \to \infty} s_n = 0$.

2 Main Results

Theorem 2.1. Let $H_1$ and $H_2$ be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator, and $A^*$ be the adjoint operator of $A$. Let $B_1 : H_1 \rightrightarrows H_1$ and $B_2 : H_2 \rightrightarrows H_2$ be two maximal monotone operator and $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be $\beta_1$ and $\beta_2$ inverse strongly monotone operators, respectively. Set $T := J^{B_2}_\lambda(I - \lambda g)$ and $U := J^{B_1}_\lambda(I - \lambda f)$ with $\lambda \in [0, 2\beta]$ where $\beta := \min\{\beta_1, \beta_2\}$. Let $u, x_0, x_1 \in H$ and $\{x_n\}$ be a sequence generated by

$$
\begin{align*}
&\begin{cases}
  w_n = x_n + \theta_n(x_n - x_{n-1}), \\
  y_n = U(I - \gamma A^*(I - T)A)w_n, \\
  x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \quad n \geq 1,
\end{cases}
\end{align*}
$$

where $\gamma \in [0, \frac{1}{L}]$ with $L$ being the spectral radius of operator $A^*A$. $\Omega$ is the solution set of SMVIP and assume that following conditions hold:

(i) Let $\{\theta_n\}$ be a sequence of real numbers chosen as

$$\theta_n = \begin{cases} 
  \min(\theta, \frac{\delta_n}{\|x_n - x_{n-1}\|}) & \text{if } x_n \neq x_{n-1}, \\
  \theta & \text{otherwise},
\end{cases}$$

where $\theta$ is a given positive number and $\{\delta_n\}$ is a positive real sequence such that $\delta_n = o(\alpha_n)$ and $\sum_{n=1}^{\infty} \delta_n < \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$. 

If $\Omega \neq \emptyset$, then $\{x_n\}$ converges strongly to $P_{\Omega}u$.

Proof. Let $z \in \Omega$. It is easy to see that

$$0 \in f(z) + B_1(z)$$

and

$$0 \in g(Az) + B_2(Az).$$

Thus, we have $-\lambda g(Az) \in \lambda B_2(Az)$ and then

$$Az - \lambda g(Az) \in Az + \lambda B_2(Az),$$

which can be equivalently written as:

$$(I - \lambda g)(Az) \in (I + \lambda B_2)(Az).$$

So,

$$J_{\lambda B_2}(I - \lambda g)(Az) = Az = T(Az).$$

On the other hand from (2.2) with the same manner, we have

$$z - \lambda f(z) \in z + \lambda B(z),$$

and then

$$J_{\lambda B_1}(I - \lambda f)(z) = z = U(z).$$

Therefore, we get

$$U(I - \gamma A^*(I - T)A)z = z.$$ 

From Lemma 1.7 we know that $U(I - \gamma A^*(I - T)A)$ is nonexpansive, thus we have

$$\|y_n - z\| = \|U(I - \gamma A^*(I - T)A)w_n - z\|$$

$$= \|U(I - \gamma A^*(I - T)A)w_n - U(I - \gamma A^*(I - T)A)z\|$$

$$\leq \|w_n - z\|. \quad (2.10)$$

From Lemma 1.6(2), we obtain

$$\|x_{n+1} - z\|^2 = \|\alpha_n u + (1 - \alpha_n)y_n - z\|^2$$

$$= \|\alpha_n(u - z) + (1 - \alpha_n)(y_n - z)\|^2$$

$$\leq \alpha_n\|u - z\|^2 + (1 - \alpha_n)\|y_n - z\|^2 - \alpha_n(1 - \alpha_n)\|y_n - u\|^2.$$ 

$$\leq \alpha_n\|u - z\|^2 + (1 - \alpha_n)\|w_n - z\|^2 - \alpha_n(1 - \alpha_n)\|y_n - u\|^2. \quad (2.11)$$
Define \( z_{n+1} = \alpha_n u + (1 - \alpha_n)U(I - \gamma A^* (I - T)A)z_n \), where \( z_0 \in H_1 \). Then, we get
\[
\|x_{n+1} - z_{n+1}\| = \|(1 - \alpha_n)y_n - (1 - \alpha_n)U(I - \gamma A^* (I - T)A)z_n\|
\]
\[
= (1 - \alpha_n)\|y_n - U(I - \gamma A^* (I - T)A)z_n\|
\]
\[
= (1 - \alpha_n)\|U(I - \gamma A^* (I - T)A)w_n - U(I - \gamma A^* (I - T)A)z_n\| 
\leq (1 - \alpha_n)\|w_n - z_n\|
\]
\[
= (1 - \alpha_n)\|x_n + \theta_n(x_n - x_{n-1}) - z_n\|
\leq (1 - \alpha_n)\|x_n - z_n\| + (1 - \alpha_n)\|x_n - x_{n-1}\|
\]
\[
\leq (1 - \alpha_n)\|x_n - z_n\| + \alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\|. \tag{2.12}
\]

By assumption (i) and Lemma 1.8, we obtain
\[
\lim_{n \to \infty} \|x_n - z_n\| = 0. \tag{2.13}
\]

We also have
\[
\|z_{n+1} - z\| \leq \alpha_n \|u - z\| + (1 - \alpha_n)\|U(I - \gamma A^* (I - T)A)z_n - z\|
\]
\[
\leq \alpha_n \|u - z\| + (1 - \alpha_n)\|z_n - z\|
\]
\[
\leq \max\{\|u - z\|, \|z_n - z\|\}
\]
\[
\leq \cdots
\]
\[
\leq \max\{\|u - z\|, \|z_0 - z\|\}. \tag{2.14}
\]

Thus, \( \{z_n\} \) is bounded, so as \( \{x_n\} \), \( \{y_n\} \) and \( \{w_n\} \).

Next, we show \( \|x_n - x_{n-1}\| \to 0 \). Let \( M = U(I - \gamma A^* (I - T)A) \), then we have
\[
\|x_{n+2} - x_{n+1}\| = \|(\alpha_{n+1} - \alpha_n)u + (1 - \alpha_{n+1})y_{n+1} - \alpha_n u - (1 - \alpha_n)y_n\|
\]
\[
= \|(\alpha_{n+1} - \alpha_n)u + (1 - \alpha_{n+1})y_{n+1} - \alpha_n u - (1 - \alpha_n)y_n\|
\]
\[
= \|(\alpha_{n+1} - \alpha_n)u + (1 - \alpha_{n+1})Mw_{n+1} - (1 - \alpha_n)Mw_n\|
\]
\[
= \|(\alpha_{n+1} - \alpha_n)u + (1 - \alpha_{n+1})M(x_{n+1} + \theta_n(x_{n+1} - x_n))
\]
\[
- (1 - \alpha_n)M(x_n + \theta_n(x_n - x_{n-1}))\|
\]
\[
= \|(\alpha_{n+1} - \alpha_n)u + (1 - \alpha_{n+1})M(x_{n+1} + \theta_n(x_{n+1} - x_n))
\]
\[
- M(x_n + \theta_n(x_n - x_{n-1})) + M(x_n + \theta_n(x_n - x_{n-1}))\|
\]
\[
\leq (1 - \alpha_{n+1})\|x_{n+1} - x_n\| + \|\alpha_{n+1} - \alpha_n\| \|u\| + (1 - \alpha_{n+1})\|\theta_{n+1}(x_{n+1} - x_n)
\]
\[
- \theta_n(x_n - x_{n-1})\| + |\alpha_{n+1} - \alpha_n| M(x_n + \theta_n(x_n - x_{n-1}))\|
\]
\[
\leq (1 - \alpha_{n+1})\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n| \|u\| + \|\theta_{n+1}(x_{n+1} - x_n)\|
\]
\[
+ |\theta_n(x_n - x_{n-1})| + |\alpha_{n+1} - \alpha_n| M(x_n + \theta_n(x_n - x_{n-1}))\|
\]
\[
\leq (1 - \alpha_{n+1})\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n| \|u\| + 2\delta_n + |\alpha_{n+1} - \alpha_n| M(w_n). \tag{2.15}
\]
Since \( \{w_n\} \) is bounded, we get \( \{M(w_n)\} \) is also bounded. Then it follows from the assumptions and Lemma 1.8 that
\[
\lim_{n \to \infty} \|x_n - x_{n-1}\| = 0. \tag{2.16}
\]
On the other hand, we have
\[
\|x_{n+1} - x_n\|^2 = \|\alpha_n(u - x_n) + (1 - \alpha_n)(y_n - x_n)\|
= \alpha_n\|u - x_n\|^2 + (1 - \alpha_n)\|y_n - x_n\|^2 - \alpha_n(1 - \alpha_n)\|y_n - u\|^2. \tag{2.17}
\]
So
\[
(1 - \alpha_n)\|y_n - x_n\|^2 = \|x_{n+1} - x_n\|^2 - \alpha_n\|u - x_n\|^2 + \alpha_n(1 - \alpha_n)\|y_n - u\|^2,
\tag{2.18}
\]
which means
\[
\lim_{n \to \infty} \|y_n - x_n\| = 0. \tag{2.19}
\]
From the definition of \(w_n\), we obtain
\[
\|x_n - w_n\| = \|\theta_n(x_n - x_{n-1})\| \to 0. \tag{2.20}
\]
Hence
\[
\lim_{n \to \infty} \|x_n - w_n\| = 0. \tag{2.21}
\]
Now, we prove \( \limsup_{n \to \infty} \langle u - u_0, y_n - u_0 \rangle \leq 0 \), where \( u_0 = P_{\Omega}u \).
Because \( \{y_n\} \) is bounded, there exists a subsequence \( \{y_{n_i}\} \) of \( \{y_n\} \) such that \( y_{n_i} \rightharpoonup \omega \in H_1 \) and \( \limsup_{n \to \infty} \langle u - u_0, y_n - u_0 \rangle = \lim_{i \to \infty} \langle u - u_0, y_{n_i} - u_0 \rangle \).
Since \( \|y_n - w_n\| \leq \|y_n - x_n\| + \|x_n - w_n\| \), we have
\[
\lim_{n \to \infty} \|y_n - w_n\| = 0. \tag{2.22}
\]
From algorithm (2.1), we know
\[
\lim_{i \to \infty} \|M(w_{n_i}) - w_{n_i}\| = \lim_{i \to \infty} \|y_{n_i} - w_{n_i}\| = 0. \tag{2.23}
\]
Since \( M \) is nonexpansive, so \( M \) is demi-closedness at zero. Thus we obtain \( M(\omega) = \omega \). Hence from Lemma 1.4 and Lemma 1.7 we get \((I - \gamma A^*(I - T)A)w = w \) and \( U(\omega) = \omega \), which means (2.6) and (2.8) hold. Equivalently (2.2) and (2.3) hold, and then \( \omega \in \Omega \). From \( u_0 = P_{\Omega}u \) and inequality (1.9), we get
\[
\limsup_{n \to \infty} \langle u - u_0, y_n - u_0 \rangle = \lim_{n \to \infty} \langle u - u_0, y_{n_i} - u_0 \rangle = \langle u - u_0, \omega - u_0 \rangle \leq 0.
\tag{2.24}
\]
By the definition of $w_n$ and Lemma 1.6(2), we have
\[
\|w_n - u_0\|^2 = \|x_n - u_0 + \theta_n(x_n - x_{n-1})\|^2 \\
\leq \|x_n - u_0\|^2 + 2\theta_n\langle w_n - z, x_n - x_{n-1}\rangle \\
\leq \|x_n - u_0\|^2 + \theta_n\rho
\]
for some $\rho > 0$. Then from Lemma 1.6(1), we get
\[
\|x_{n+1} - u_0\|^2 = \|\alpha_n(u - u_0) + (1 - \alpha_n)(y_n - u_0)\|^2 \\
\leq (1 - \alpha_n)^2\|y_n - u_0\|^2 + 2\alpha_n\langle u - u_0, x_{n+1} - u_0\rangle \\
\leq (1 - \alpha_n)^2\|y_n - u_0\|^2 + 2\alpha_n\langle u - u_0, x_{n+1} - u_0\rangle \\
\leq (1 - \alpha_n)\|w_n - u_0\|^2 + 2\alpha_n\langle u - u_0, x_{n+1} - u_0\rangle \\
\leq (1 - \alpha_n)\|x_n - u_0\|^2 + \theta_n\rho + 2\alpha_n\langle u - u_0, x_{n+1} - x_n - x_n - y_n + y_n - u_0\rangle \\
= (1 - \alpha_n)\|x_n - u_0\|^2 + \alpha_n\left(\frac{\theta_n}{\alpha_n}\rho + 2\langle u - u_0, x_{n+1} - x_n\rangle\right) \\
+ 2\langle u - u_0, x_n - y_n\rangle + 2\langle u - u_0, y_n - u_0\rangle
\]
(2.26)
Combining (2.26) with (2.16), (2.19), (2.24) and using Lemma 1.8, we obtain
\[
\lim_{n \to \infty} \|x_n - u_0\| = 0.
\]
(2.27)
Hence $\{x_n\}$ converges strongly to $u_0 = P_\Omega u$. The proof is completed. \hfill \Box

Remark. We note that the condition (i) is easily implemented in numerical computation since the value of $\|x_n - x_{n-1}\|$ is known before choosing $\theta_n$ at each step.

If $f \equiv 0$ and $h \equiv 0$, SMVIP reduces to the split variational inclusion problem (shortly, $SVIP$) written as
\[
\text{finding } x^* \in H_1 \text{ such that } 0 \in B_1(x^*) \text{ and } 0 \in B_2(Ax^*),
\]
where $A : H_1 \to H_2$ is a bounded linear operator, $B_1$ and $B_2$ are two maximal monotone operator. And the solution set of $SVIP$ is denoted by
\[
\Gamma = \{x^* : 0 \in B_1(x^*) \text{ and } 0 \in B_2(Ax^*)\}.
\]
Using Theorem 2.1, we obtain the following result immediately.

Corollary 2.2. Let $H_1$ and $H_2$ be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator, and $A^*$ be the adjoint operator of $A$. Let $B_1 : H_1 \to H_1$ and $B_2 : H_2 \to H_2$ be two maximal monotone operator. Let $u, x_0, x_1 \in H$, and $\{x_n\}$ be a sequence generated by
\[
\begin{cases} 
  w_n = x_n + \theta_n(x_n - x_{n-1}), \\
  y_n = J_{B_1}^\gamma(I - \gamma A^*(I - J_{B_2}^\gamma)A)w_n, \\
  x_{n+1} = \alpha_n u + (1 - \alpha_n y_n), \quad n \geq 1,
\end{cases}
\]
where $\gamma \in [0, \frac{1}{L}]$ with $L$ being the spectral radius of operator $A^*A$. Assume that following conditions hold:

(i) Let $\{\theta_n\}$ be a sequence of real numbers chosen as

$$\theta_n = \begin{cases} 
\min(\theta, \frac{\delta_n}{\|x_n - x_{n-1}\|}) & \text{if } x_n \neq x_{n-1}, \\
\theta & \text{otherwise,}
\end{cases}$$

where $\{\delta_n\}$ is a positive real sequence such that $\delta_n = o(\alpha_n)$, $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\theta > 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.

If $\Gamma \neq 0$, then $\{x_n\}$ converges strongly to $P_\Gamma u$.

3 Numerical examples

In this section, we present a numerical example to compare the convergence performance of our algorithm with the algorithm proposed by Moudafi [7] for solving $\text{SMVIP}$ in $\mathbb{R}^2$. All the codes are written in the Matlab2016a.

Example 3.1. Let $H_1 = H_2 = \mathbb{R}^2$, and $A,B_1,B_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be $A := \begin{pmatrix} 7 & 9 \\ 2 & 1 \end{pmatrix}$, $B_1 := \begin{pmatrix} 8 & 0 \\ 0 & 5 \end{pmatrix}$ and $B_2 := \begin{pmatrix} 7 & 0 \\ 0 & 6 \end{pmatrix}$, respectively and $f_1,f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be $f_1 = f_2 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}$. Find a point $\bar{x} = (\bar{x}_1, \bar{x}_2)^T \in \mathbb{R}^2$ such that $f_1(\bar{x}) + B_1(\bar{x}) = (0,0)^T$ and $f_2(A\bar{x}) + B_2(A\bar{x}) = (0,0)^T$. Indeed, $\bar{x}_1 = \bar{x}_2 = 0$. let $\epsilon_n := \|x_n - \bar{x}\|$ be the error term. We choose initial values $x_0 = (10,10)^T$ and $x_1 = (10,10)^T$, $\gamma = 0.5/\|A\|^2$, $\alpha_n = 1/n$, $u = (1,1)^T$,

$$\theta_n = \begin{cases} 
\min\{0.5, \frac{1}{n\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\
0.5 & \text{otherwise.}
\end{cases}$$

The perform of numerical results are shown in Figure 1 and Table 1.

<table>
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<tr>
<th>n</th>
<th>$\epsilon_n$ by (2.1)</th>
<th>$\epsilon_n$ by (1.5)</th>
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<td>2.81524428639800</td>
</tr>
</tbody>
</table>
Remark 3.2. In numerical example, it is showed that Algorithm (2.1) has faster convergence rate than the algorithm (1.5) of Moudafi [2].

References


[6] Censor, Y., Bortfeld, T., Martin, B., A. Trofimov, Aunified approach for inversion problems in intensity modulated radiation therapy,
[1] Huang Shuang and Lin Wang


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