S \mathcal{I} F\text{-ring}

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Abstract

A ring \mathcal{R} is called right simple \mathcal{I}\text{-flat} (or right S\mathcal{I}F\text{-ring for short) if, for every simple right \mathcal{R} -module is \mathcal{I}\text{-flat}. In this paper, several properties of this class of rings are given, some interesting results are obtained. Right S\mathcal{I}F\text{-rings, } \mathcal{K}\mathcal{S} rings are studied, several conditions under which right S\mathcal{I}F\text{-rings are reduced rings, strongly } \mathcal{V}\text{-regular rings are given.

Keywords: } \mathcal{I}\text{-flat, } \mathcal{I}\text{-regular ring, } \mathcal{K}\mathcal{S} ring, strongly } \mathcal{V}\text{-regular ring

1 Introduction

Throughout this paper, \mathcal{R} will be an associative ring with identity and all modules are unitary right \mathcal{R} -modules. For \sigma \in \mathcal{R}, r(\sigma), l(\sigma) denoted the right annihilator and the left annihilator of \sigma, respectively. We write \mathcal{Y}(\mathcal{R}) (\mathcal{Z}(\mathcal{R})), \mathcal{K}(\mathcal{R}), \mathcal{I}(\mathcal{R}), \mathcal{S} for the right (left) singular ideal, the set of nilpotent element, Jacobson radical of \mathcal{R} and Tansor product, respectively. A ring \mathcal{R} is reversible iff \(r(\sigma)l(\sigma)) is an ideal of \mathcal{R}, for every \sigma \in \mathcal{R} [3].

Generalization of flat have been discussed in many papers (see [4], [7], [10]). \mathcal{R} is right (left) SF\text{-ring ([7]), if simple right (left) } \mathcal{R}\text{-modules are flat. In ([10], [6]), S\mathcal{N}F\text{-rings are defined and studied. A ring } \mathcal{R} \text{ is called right (left) S\mathcal{N}F\text{-ring, if simple right (left) } \mathcal{R}\text{-modules are } \mathcal{K}\text{-flat. The nice structure of S\mathcal{N}F\text{-rings draws our attention to define S\mathcal{I}F\text{-ring and to investigate } \mathcal{I}\text{-regular rings, strongly } \mathcal{V}\text{-regular rings and reduced rings.

As usual, \mathcal{R} is a reduced ring if } \mathcal{K}(\mathcal{R}) = 0 [7]. \mathcal{R} is called } \mathcal{K}\mathcal{S}, \text{ if } \mathcal{K}(\mathcal{R}) \subseteq \mathcal{I}(\mathcal{R}) [1]. A ring \mathcal{R} is called } \mathcal{I}\text{-regular ring [12], if } \sigma \in \sigma \mathcal{R} \sigma \text{ for all } \sigma \in \mathcal{I}(\mathcal{R}). \text{ Chen, [2] called a ring } \mathcal{R} \text{ weakly semi-commutative, if } \sigma b = 0 \text{ implies } \sigma \mathcal{R} b \subseteq \mathcal{K}(\mathcal{R}) \text{ for } \sigma, b \in \mathcal{R}.
2 S3F-ring

Following [12], a right \(\mathcal{R}\)-module \(M\) is called \(\mathfrak{I}\)-flat, if for any \(\sigma \in \mathfrak{I}(\mathcal{R})\), the map \(i_M \otimes i: M \otimes_{\mathcal{R}} \mathcal{R}\sigma \to M \otimes \mathcal{R}\) is monic.

Now we give the following definition:

**Lemma 2.1:** If \(I\) is a right ideal of \(\mathcal{R}\). Then \(\mathcal{R}/I\) is \(\mathfrak{I}\)-flat module iff \(\sigma = I \cap \mathcal{R}\sigma\), for every \(\sigma \in \mathfrak{I}(\mathcal{R})\).

**Proof:** It can be proved by same method as [10].

**Definition 2.2:** A ring \(\mathcal{R}\) is said to be right (left) S3F-ring, if every simple right (left) \(\mathcal{R}\)-module is \(\mathfrak{I}\)-flat.

Every right SF-ring is right S3F-ring, but the converse is not true.

**Examples:**
1. The rings \(\mathcal{Z}_p\) and \(\mathcal{Z}_{pq}\) where \(p\) and \(q\) are prime numbers are S3F-ring.
2. The ring of integer \(\mathcal{Z}\) is S3F-ring, but it is not SF-ring.

**Proposition 2.3:** Let \(\mathcal{R}\) be a right S3F-ring. Then \(\mathcal{R} = \mathcal{M} + \ell(\sigma)\), where \(\mathcal{M}\) is a maximal right ideal of \(\mathcal{R}\) and \(\sigma \in \mathfrak{I}(\mathcal{R})\).

**Proof:** Since \(\mathcal{R}\) be right S3F-ring, then \(\mathcal{R}/\mathcal{M}\) is \(\mathfrak{I}\)-flat ring. Thus by (Lemma 2.1), for all \(\sigma \in \mathfrak{I}(\mathcal{R}) \subseteq \mathcal{M}\), there exists \(b \in \mathcal{M}\) such that \(\sigma = b\sigma\). Hence \((1 - b) \in \ell(\sigma)\) and \(1 = b + (1 - b)\). Therefore \(\mathcal{R} = \mathcal{M} + \ell(\sigma)\).

**Proposition 2.4:** If \(\mathcal{R}\) is right S3F-ring. Then every left nonzero divisor of \(\mathcal{R}\) is invertible.

**Proof:** Let \(0 \neq \sigma \in \mathcal{R}\) be a nonzero divisor of \(\mathcal{R}\) and let \(\sigma\mathcal{R} \neq \mathcal{R}\). Choose a right maximal ideal \(K\) respect to \(\sigma\mathcal{R} \subseteq K\). Since \(\mathcal{R}/K\) is \(\mathfrak{I}\)-flat ring, then there exists \(b \in K\) such that \(\sigma = b\sigma\). Hence \((1 - b) \in \ell(\sigma)\) and \(1 = b + (1 - b)\), a contradiction. So \(\sigma\mathcal{R} = \mathcal{R}\) and hence it is a right invertible.

**Theorem 2.5:** If \(\mathcal{R}\) is right S3F-ring with \(\ell(\sigma) = 0\) for all \(\sigma \in \mathfrak{I}(\mathcal{R})\). Then \(\mathfrak{I}(\mathcal{R}) = 0\).

**Proof:** Since \(\mathcal{R}/\mathcal{M}\) is \(\mathfrak{I}\)-flat ring, for any maximal ideal \(\mathcal{M}\) of \(\mathcal{R}\), then for all \(\sigma \in \mathfrak{I}(\mathcal{R})\), there exists \(b \in \mathcal{M}\) such that \(\sigma = b\sigma\). Hence \((1 - b) \in \ell(\sigma) = 0\), implies that \(1 \in \mathcal{M}\) a contradiction. Thus we have \(\sigma = 0\). Therefore \(\mathfrak{I}(\mathcal{R}) = 0\).

**Corollary 2.6:** If \(\mathcal{R}\) is right S3F-ring with \(\ell(\sigma) = 0\) for all \(\sigma \in \mathfrak{I}(\mathcal{R})\). Then \(\mathcal{Y}(\mathcal{R}) \cap \mathfrak{I}(\mathcal{R}) = 0\).
From (Theorem 2.5) and (proposition 2.19, [2]) we get the corollary:

**Corollary 2.7:** If $\mathcal{R}$ is right $\mathfrak{S}\mathfrak{F}$-ring and weakly semi commutative with $\ell(\sigma) = 0$ for all $\sigma \in \mathfrak{S}(\mathcal{R})$. Then $\mathcal{R}$ is reduced.

**Proposition 2.8:** Let $\mathcal{R}$ be a ring. Then $\mathcal{R}/\mathfrak{S}(\mathcal{R})$ is $\mathfrak{S}$-flat iff $\mathfrak{S}(\mathcal{R}) = 0$.

**Proof:** Assume $\mathcal{R}/\mathfrak{S}(\mathcal{R})$ is $\mathfrak{S}$-flat, then by (Lemma 2.1) for any $x \in \mathfrak{S}(\mathcal{R})$, there exists $y \in \mathfrak{S}(\mathcal{R})$ such that $x = xy$. Hence $(1 - y)x = 0$. Since $x \in \mathfrak{S}(\mathcal{R})$, $(1 - y)$ is invertible, and so $(1 - y)v = 1$, for some $v \in \mathcal{R}$. Hence $x = 0$, and so $\mathfrak{S}(\mathcal{R}) = 0$. The converse is trivial.

3 Regularity of right $\mathfrak{S}\mathfrak{F}$-ring

$\mathcal{R}$ is called right (left) $\mathfrak{S}$-weakly regular ring [5], if for every $\sigma \in \mathfrak{S}(\mathcal{R})$, then $\sigma \in \sigma \mathcal{R} \sigma \mathcal{R}$ ($\sigma \in \mathcal{R} \sigma \mathcal{R} \sigma$).

**Lemma 3.1:** Let $\mathcal{R}$ be a right nonsingular and $r(\sigma) \subseteq \ell(\sigma)$, for every $\sigma \in \mathcal{R}$. Then $\mathcal{R}$ is reduced. [9]

Following [12], a ring $\mathcal{R}$ is called a right (left) $\mathfrak{S}$PP-ring, if for any $\sigma \in \mathfrak{S}(\mathcal{R})$, $\sigma \mathcal{R}(\mathfrak{S} \mathcal{R})$ is projective.

**Proposition 3.2:** If $\mathcal{R}$ be right $\mathfrak{S}$PP-ring with $r(\sigma) \subseteq \ell(\sigma)$, for every $\sigma \in \mathfrak{S}(\mathcal{R})$, then $\mathcal{R}$ reduced ring.

**Proof:** Let $0 \neq \sigma \in Y(\mathcal{R})$ with $\sigma^2 = 0$. Since $\mathcal{R}$ is right $\mathfrak{S}$PP-ring, $\sigma \mathcal{R}$ is projective. So $r(\sigma)$ is direct summand of $\mathcal{R}$ as a right $\mathcal{R}$-module. But $\sigma \in Y(\mathcal{R})$, $r(\sigma)$ must be essential in $\mathcal{R}$, which is contradiction. Hence $Y(\mathcal{R}) = 0$. By (Lemma 3.1) we get $\mathcal{R}$ reduced ring.

**Theorem 3.3:** If $\mathcal{R}$ is right $\mathfrak{S}\mathfrak{F}$-ring, $\mathfrak{S}$PP-ring and $\ell(\sigma) \subseteq r(\sigma)$ for every $\sigma \in \mathfrak{S}(\mathcal{R})$, then $\mathcal{R}$ is $\mathfrak{S}$-weakly regular ring.

**Proof:** We will show that $\mathcal{R} \sigma \mathcal{R} + r(\sigma) = \mathcal{R}$, for any $\sigma \in \mathfrak{S}(\mathcal{R})$. Suppose that $\mathcal{R}b \mathcal{R} + r(b) \neq \mathcal{R}$, for any $b \in \mathfrak{S}(\mathcal{R})$, then there exists a maximal right ideal $\mathcal{M}$ containing $\mathcal{R}b \mathcal{R} + r(b)$. Since $\mathcal{R}$ is right $\mathfrak{S}\mathfrak{F}$-ring, then as $\mathcal{R}/\mathcal{M}$ is $\mathfrak{S}$-flat, and there exists $x \in \mathcal{M}$ such that $b = xb$ (Lemma 2.1). Hence $1 - x \in \ell(b) = r(b) \subseteq \mathcal{M}$, (Proposition 3.2), and $1 \in \mathcal{M}$. This is a contradiction. Therefore $\mathcal{R} \sigma \mathcal{R} + r(\sigma) = \mathcal{R}$. Hence we can write $d \sigma b + z = 1$, for some $z \in r(\sigma)$ and, $b \in \mathcal{R}$. Since $\sigma z = 0$, this gives $\sigma = \sigma d \sigma b$. Thus $\mathcal{R}$ is right $\mathfrak{S}$-weakly regular ring.

While every $\mathfrak{S}$-regular ring is right and left $\mathfrak{S}\mathfrak{F}$-ring, we do not know whether the converse is true. However we have:
**Proposition 3.4:** Let \( \mathcal{R} \) be a ring such that the left annihilator of any element of \( \mathfrak{I}(\mathcal{R}) \) is also a right ideal. If \( \mathcal{R} \) is a right \( \mathfrak{S}\mathfrak{F} \)-ring, then \( \mathcal{R} \) is \( \mathfrak{I} \)-regular ring.

**Proof:** Let \( \mathcal{R} \) is right \( \mathfrak{S}\mathfrak{F} \)-ring and \( z \in \mathfrak{I}(\mathcal{R}) \). If \( K \) is left annihilator of \( z \) in \( \mathfrak{I}(\mathcal{R}) \), then \( K \) is also a right ideal by hypothesis. If \( \mathcal{M} \) is a maximal right ideal of \( \mathcal{R} \) containing \( z\mathcal{R} + K \), since \( \mathcal{R} \) is right \( \mathfrak{S}\mathfrak{F} \)-ring, then as \( \mathcal{R}/\mathcal{M} \) is \( \mathfrak{I} \)-flat, \( z = \sigma z \), for some \( \sigma \in \mathcal{M} \) (Lemma 2.1). Hence \( (1 - \sigma) \in K \subset \mathcal{M} \), and \( 1 \in \mathcal{M} \), a contradiction. Therefore \( z\mathcal{R} + K = \mathcal{R} \), and \( zb + k = 1 \), for some \( b \in \mathcal{R} \) and \( k \in K \). Since \( z = 0 \), this gives \( z = zbz \). Thus \( \mathcal{R} \) is \( \mathfrak{I} \)-regular ring.

**Theorem 3.5:** Let \( \mathcal{R} \) be a ring and \( \sigma\mathcal{R} \) is maximal right ideal of \( \mathcal{R} \) for all \( \sigma \in \mathfrak{I}(\mathcal{R}) \). Then \( \mathcal{R} \) is \( \mathfrak{I} \)-regular ring iff \( \mathcal{R} \) is right \( \mathfrak{S}\mathfrak{F} \)-ring.

**Proof:** Assume that \( \mathcal{R} \) is right \( \mathfrak{S}\mathfrak{F} \)-ring and \( \sigma\mathcal{R} \) is maximal right ideal for all \( \sigma \in \mathfrak{I}(\mathcal{R}) \). Then \( \mathcal{R}/\sigma\mathcal{R} \) is simple right \( \mathcal{R} \)-module. Thus \( \mathcal{R}/\sigma\mathcal{R} \) is \( \mathfrak{I} \)-flat, \( \sigma = x\sigma \), for some \( x \in \sigma\mathcal{R} \). Hence \( \sigma = x\sigma = \sigma b\sigma \), for some \( b \in \mathcal{R} \). Thus \( \mathcal{R} \) is \( \mathfrak{I} \)-regular ring. The converse is trivial.

Recall that a ring \( \mathcal{R} \) is called right (left) strongly \( \mathcal{Y} \)-regular ring [8], if for every element \( \sigma \in \mathcal{R} \), there exists \( b \in \mathcal{R} \) and \( 1 \neq n \in \mathbb{Z}^+ \) such that \( \sigma = \sigma^2b^n (\sigma = b^n\sigma^2) \). A ring \( \mathcal{R} \) is called strongly \( \mathcal{Y} \)-regular ring if it is both right and left strongly \( \mathcal{Y} \)-regular ring. [8]

**Theorem 3.6:** Let \( \mathcal{R} \) be right \( \mathfrak{S}\mathfrak{F} \)-ring and \( \mathcal{R}\sigma = \mathcal{R}\sigma^2 \), for all \( \sigma \in \mathfrak{I}(\mathcal{R}) \). Then \( \mathcal{R} \) is left strongly \( \mathcal{Y} \)-regular ring.

**Proof:** Since \( \mathcal{R} \) is right \( \mathfrak{S}\mathfrak{F} \)-ring, then \( \mathcal{R}/\mathcal{M} \) is \( \mathfrak{I} \)-flat and every \( \sigma \in \mathfrak{I}(\mathcal{R}) \subset \mathcal{M} \), there exists \( b \in \mathcal{M} \) such that \( \sigma = b\sigma = bb\sigma = b^2\sigma = \cdots = b^n\sigma \), \( n \in \mathbb{Z}^+ \). Thus \( \sigma = b^n\sigma \in \mathcal{R}\sigma = \mathcal{R}\sigma^2 \). Therefore \( \sigma = b^n\sigma^2 \) and so \( \mathcal{R} \) is left strongly \( \mathcal{Y} \)-regular ring.

**Corollary 3.7:** Let \( \mathcal{R} \) be reversible, right \( \mathfrak{S}\mathfrak{F} \)-ring and \( \mathcal{R}\sigma = \mathcal{R}\sigma^2 \), for all \( \sigma \in \mathfrak{I}(\mathcal{R}) \). Then \( \mathcal{R} \) is strongly \( \mathcal{Y} \)-regular ring.

4 Some result on \( \mathfrak{N}\mathfrak{I} \) rings

Following [10], a ring \( \mathcal{R} \) is called right (left) \( \mathfrak{S}\mathfrak{N}\mathfrak{F} \)-rings, if every simple right (left) \( \mathcal{R} \)-module is \( \mathfrak{K} \)-flat. Every reduced ring is \( \mathfrak{K} \)-flat.

**Proposition 4.1:** If \( \mathcal{R} \) is \( \mathfrak{N}\mathfrak{I} \) right \( \mathfrak{S}\mathfrak{F} \)-ring and \( \ell(\sigma) \subseteq \mathfrak{r}(\sigma) \), for all \( \sigma \in \mathfrak{I}(\mathcal{R}) \). Then \( \mathcal{R} \) is reduced.

**Proof:** Let \( \mathcal{R} \) is not reduced, then there exists \( 0 \neq \sigma \in \mathcal{R} \) such that \( \sigma^2 = 0 \). Since \( \sigma \neq 0 \), then there exists a maximal right ideal \( \mathcal{M} \) of \( \mathcal{R} \) containing \( \mathfrak{r}(\sigma) \). Since \( \mathcal{R} \)
is right S3F-ring, then $\mathcal{R}/\mathcal{M}$ is $\mathcal{I}$-flat and by (Lemma 2.1), $\sigma = b\sigma$, for some $b \in \mathcal{M}$. Thus $(1 - b) \in \ell(\sigma) \subseteq r(\sigma) \subseteq \mathcal{M}$, and so $1 \in \mathcal{M}$, a contradiction. Therefore $\sigma = 0$, and $\mathcal{R}$ is reduced.

Now, we give relation between SRF-ring and S3F-ring:

**Corollary 4.2:** If $\mathcal{R}$ is $\mathcal{N}$ and S3F-ring and $\ell(\sigma) \subseteq r(\sigma)$, for all $\sigma \in \mathcal{I}(\mathcal{R})$. Then $\mathcal{R}$ is SRF-ring.

**Proof:** From (Proposition 4.1).

**Proposition 4.3:** Let $\mathcal{R}$ be $\mathcal{N}$ ring and $\mathcal{I}(\mathcal{R}\sigma\mathcal{R}) = \mathcal{N}(\mathcal{R}\sigma\mathcal{R})$, for every $\sigma \in \mathcal{I}(\mathcal{R})$. Then $\mathcal{R}$ is right S3F-ring iff $\mathcal{R}$ is right SRF-ring.

**Proof:** Since $\mathcal{R}$ be $\mathcal{N}$ ring, then $\mathcal{N}(\mathcal{R}) \subseteq \mathcal{I}(\mathcal{R})$. Now we show that $\mathcal{I}(\mathcal{R}) \subseteq \mathcal{N}(\mathcal{R})$. If $\mathcal{I}(\mathcal{R}) \subseteq \mathcal{N}(\mathcal{R})$. Then $\mathcal{I}(\mathcal{R}\mathcal{R}) = \mathcal{N}(\mathcal{R}\mathcal{R})$ by the hypothesis. Since $b \in \mathcal{R}\mathcal{R}$ and $\in \mathcal{I}(\mathcal{R})$, then $b \in \mathcal{I}(\mathcal{R}) \cap (\mathcal{R}b\mathcal{R}) = \mathcal{I}(\mathcal{R}b\mathcal{R})$, and so $b \in \mathcal{I}(\mathcal{R}b\mathcal{R}) = \mathcal{N}(\mathcal{R}) \cap (\mathcal{R}b\mathcal{R})$. Thus $b \in \mathcal{N}(\mathcal{R})$ and $\mathcal{N}(\mathcal{R}) = \mathcal{I}(\mathcal{R})$. Therefore $\mathcal{R}$ is right S3F-ring iff $\mathcal{R}$ is right SRF-ring.

**Theorem 4.4:** Let $\mathcal{R}$ be $\mathcal{N}$ ring and $\ell(\sigma) \subseteq r(\sigma)$, for all $\sigma \in \mathcal{I}(\mathcal{R})$, whose every simple singular right $\mathcal{R}$-module is $\mathcal{I}$-flat. Then $\mathcal{Y}(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R}) = 0$.

**Proof:** If $\mathcal{Y}(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R}) \neq 0$, then there exists $0 \neq b \in \mathcal{Y}(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R})$, such that $b^2 = 0$. We claim that $\mathcal{R}b\mathcal{R} + r(\mathcal{b}) = \mathcal{R}$. Otherwise there exists a maximal essential right ideal $\mathcal{M}$ of $\mathcal{R}$ containing $\mathcal{R}b\mathcal{R} + r(\mathcal{b})$. So $\mathcal{M}/\mathcal{M}$ simple singular right $\mathcal{R}$-module, and then it is right $\mathcal{I}$-flat. Since $\mathcal{R}$ is $\mathcal{N}$ ring then $b \in \mathcal{I}(\mathcal{R})$ and $b = xb$ for some $x \in \mathcal{M}$. Since $(1 - b) \in \ell(x) \subseteq r(x) \subseteq \mathcal{M}$, then $1 \in \mathcal{M}$, which is a contradiction. Therefore $1 \in \mathcal{M}$, $\mathcal{I}$-flat, which is a contradiction. We claim that $\mathcal{Y}(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R}) = 0$.

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**References**


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