Inertial Algorithm for $J$-Equilibrium Problem in Banach Spaces

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Abstract

In this paper, a new iterative method to approximate a solution of $J$-equilibrium problem in the dual spaces of 2-uniformly convex and 2-uniformly smooth Banach spaces is introduced by combining proximal regularized technique with inertial method, and the strong convergence theorem of the iterative scheme presented is obtained.

Keywords: $J$-equilibrium problem; inertial algorithm; proximal regularized technique; convergence

1 Introduction

Equilibrium theory provides a powerful framework for studying many nonlinear problems arising from finance, economics, transportation, mechanics and optimization. Many practical problems in the fields of investment decision-making, traffic analysis and signal professing also can be attributed to the equilibrium problem [1–8]. Moreover, the equilibrium problem provides a unified form for minimization problem, saddle point problem, variational inequality problem, fixed point problems and complementarity problem.

Mathematically, the equilibrium problem (shortly, $EP$) in Banach space can be stated as follows: assume that $E$ is a real Banach space, and $C$ is a

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nonempty closed convex subset of $E$, $f$ is a bifunction from $C \times C$ to $R$. $EP$ is to find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C. \quad (1.1)$$

For approximating a solution of $EP(1.1)$, many iterative algorithms have been proposed, see [9–23] and the references therein. Among them, using proximity mapping to solve $EP$ is a commonly used method in recent years. The proximity mapping $prox_{\lambda f}: C \times C \to R$ is defined by the following:

$$\arg\min \{\lambda f(x, y) + \frac{1}{2}\|y - x\|^2 : y \in C\}.$$ where $f : C \times C \to R$ is a function, and $f(x, \cdot)$ be proper, convex, lower semicontinuous and subdifferentiable for all $x \in C$. $\lambda > 0$ is a constant.

To speed up the convergence rate, Polyak [24] proposed an inertial extrapolation to act as an acceleration process to solve the problem of smooth convex minimization based on the heavy ball methods of a second-order time dynamic system. Recently, some researchers have constructed some faster iterative algorithms by using inertial extrapolation. These include inertial proximal point methods [25], inertial forward-backward methods [26], etc. But the following condition usually is needed to obtain convergence of the iteration scheme

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < +\infty. \quad (1.2)$$

In 2008, inspired by the results of Ibaraki [32], and Takahashi et al [33], Takahashi and Zembayashi [34] introduced $J$-equilibrium problem (shortly, $JEP$) in the dual spaces of Banach spaces as follows:

Let $E$ be a smooth Banach space with the dual space $E^*$, and $C$ be a nonempty closed convex subset of $E$ such that $JC$ is a closed and convex subset of $E^*$, where $J$ is the normalized duality mapping from $E$ onto $E^*$. Let $f : JC \times JC \to R$ be a bifunction, find $\bar{x} \in C$ such that

$$f(J\bar{x}, Jy) \geq 0, \quad \forall y \in C. \quad (1.3)$$

The solutions set of $JEP$ is defined by $JEP(f, C)$. At present, some methods have been proposed to solve $JEP$, see, for instance, [35–37].

In 2018, by combining proximal mapping with inertial algorithm, Dang [27] constructed an algorithm to solve a class of $EP$ in Hilbert spaces, and obtained the strong convergence and linear convergence theorem without the condition (1.2).
In this paper, motivated by the above works and related literatures [28–31], combining the proximal regularized technique with inertial method, we introduce a new algorithm to solve JEP in 2-uniformly convex and 2-uniformly smooth Banach space, and obtain a strong convergence theorem without the condition (1.2).

2 Preliminaries

Let $E$ be a real Banach space with the dual space $E^*$. A Banach space $E$ is said to be strictly convex if $\|x+y\| \leq 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. The modulus of convexity of $E$ is defined by

$$\delta_E(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\},$$

for all $\varepsilon \in [0, 2]$. $E$ is said to be uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\varepsilon) > 0$ for all $0 < \varepsilon < 2$.

Let $\rho_E : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of $E$ defined by

$$\rho_E(t) = \sup\{\|x+y\| + \|x-y\| - 1 : x \in U, \|y\| \leq t\}.$$

A Banach space $E$ is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$.

Let $p$ be a fixed real number with $p > 1$, then a Banach space $E$ is called to be $p$-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\varepsilon) \geq c \varepsilon^p$. Let $q$ be a fixed real number with $q > 1$, then a Banach space $E$ is called to be $q$-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q$ for all $t > 0$. It is well known that if $E$ is $p$-uniformly convex, then $E^*$ is $q$-uniformly smooth, if $E$ is $q$-uniformly smooth, then $E^*$ is $p$-uniformly convex.

The normalized duality mapping $J$ from $E$ into $E^*$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \forall x \in E.$$

If $E$ is smooth, then $J$ is single-valued. If $E$ is reflexive, smooth and strictly convex Banach space and $J^* : E^* \to 2^E$ is the normalized duality mapping on $E^*$, then $J^{-1} = J^*$. If $E$ is uniformly convex and uniformly smooth, then $J$ is uniformly norm-to-norm continuous on bounded sets of $E$.

Let $E$ be a smooth, strictly convex and reflexive Banach space, define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2,$$

for all $x, y \in E$, which is called Lyapunov functional and has the following properties.
\( (\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2. \)  
(2.2)

\( \phi(x, y) = \phi(x, z) + \phi(z, y) + 2(x - z, Jz - Jy). \)  
(2.3)

\[ 2(x - y, Jz - Jw) = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w). \]  
(2.4)

It is worth noting that \( \phi(x, y) = 0 \iff x = y \) and if \( E \) is a real Hilbert space, then \( \phi(x, y) = \|x - y\|^2 \).

In [38], let \( E \) be a reflexive, strictly convex and smooth Banach space, they defined

\[ \phi_s(x^*, y^*) = \|x^*\|^2 - 2\langle x^*, Jx \rangle \]  

for \( x^*, y^* \in E^* \). As can be seen, this is a Lyapunov functional on \( E^* \), and easy to know \( \phi(x, y) = \phi_s(Jy, Jx) \).

If \( C \) is a convex subset of a Banach space \( E \), the normal cone for \( C \) at a point \( v \in C \) is \( N_C(v) = \{ x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \forall y \in C \} \).

Suppose that \( E \) is a Banach space and let \( f : E \to (-\infty, +\infty] \) be a function. For \( x_0 \in Dom(f) \), the subdifferential of \( f \) at \( x_0 \) is defined as

\[ \partial f(x_0) = \{ x^* \in E^* : f(x) \geq f(x_0) + \langle x^*, x - x_0 \rangle, \forall x \in E \}. \]

If \( \partial f(x_0) \neq \emptyset \), then \( f \) is subdifferentiable at \( x_0 \).

**Lemma 2.1** [39] Let \( E \) be a 2-uniformly smooth Banach space with the best smoothness constants \( k > 0 \), then the following inequality holds:

1. \( \|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx \rangle + 2\|ky\|^2 \) for all \( x, y \in E \);
2. for any \( x, y \in E \), \( \|x + y\|^2 \geq \|x\|^2 + 2\langle y, Jx \rangle \).

**Lemma 2.2** [40] Let \( E \) be a uniformly convex Banach space and \( r > 0 \), then, there exists a strictly increasing, continuous and convex function \( g : [0, 2r] \to [0, +\infty) \) such that \( g(0) = 0 \) and

\[ \|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|) \]

for all \( \lambda \in [0, 1] \) and \( x, y \in B_r = \{ z \in E : \|z\| \leq r \} \).

**Lemma 2.3** [40] Let \( E \) be a uniformly convex Banach space and \( r > 0 \), then, there exists a strictly increasing, continuous and convex function \( g : [0, 2r] \to [0, +\infty) \) such that \( g(0) = 0 \) and

\[ g(\|x - y\|) \leq \phi(x, y) \]

for all \( \lambda \in [0, 1] \) and \( x, y \in B_r = \{ z \in E : \|z\| \leq r \} \).
Lemma 2.4 [41] Let $C$ be a nonempty convex subset of a Banach space $E$ and $f : C \to R \cup \{+\infty\}$ be a convex and subdifferentiable and lower semicontinuous function. If the function $f$ satisfies the following condition:

Either $\text{int}(C) \neq \emptyset$ or $f$ is continuous at a point in $C$,

then $x^*$ is a solution to the following convex optimization problem $\min \{ f(x) : x \in C \}$ if and only if

$$0 \in \partial f(x^*) + N_C(x^*).$$

where $\partial f(x^*)$ is the subdifferential of $f$ and $N_C(x^*)$ is the normal cone of $C$ at $x^*$.

Lemma 2.5 [41] Let $E$ be a reflexive Banach space. If $f : E \to R \cup \{+\infty\}$ and $g : E \to R \cup \{+\infty\}$ are nontrivial, convex and lower continuous functions and if $0 \in \text{Int}(\text{Dom} f - \text{Dom} g)$, then

$$\partial (f + g)(x) = \partial f(x) + \partial g(x).$$

Lemma 2.6 [41] Let $E$ be a reflexive Banach space. If the convex function $f : E \to R \cup \{+\infty\}$ is continuous in the domain, then for every $x \in \text{Int}(\text{Dom} f)$, $\partial f(x)$ is nonempty and closed.

Lemma 2.7 [42] Let $C$ be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space $E$ such that $JC$ is closed and convex. Let $f$ be a bifunction from $JC \times JC$ to $R$ satisfying the following conditions:

(A1) $f(x^*, x^*) = 0$ for all $x^* \in JC$;

(A2) $f$ is monotone, i.e. $f(x^*, y^*) + f(y^*, x^*) \leq 0$ for all $x^*, y^* \in JC$;

(A3) for all $x^*, y^*, z^* \in JC$, $\limsup_{t \to 0} f(tz^* + (1 - t)x^*, y^*) \leq f(x^*, y^*)$;

(A4) for all $x^* \in JC$, $f(x^*, \cdot)$ is convex and lower semicontinuous.

then for $r > 0$ and $x \in E$, there exists $z \in C$ such that

$$f(Jz, Jy) + \frac{1}{r} \langle z - x, Jy - Jz \rangle \geq 0, \forall y \in C.$$

Lemma 2.8 [43] Let $C$ be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space $E$ such that $JC$ is closed and convex. Let $f$ be a bifunction from $JC \times JC$ to $R$ satisfying (A1) – (A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:

$$T_r(x) = \{ z \in C : f(Jz, Jy) + \frac{1}{r} \langle z - x, Jy - Jz \rangle \geq 0, \forall y \in C \}.$$

Then the following statements hold:
(i) $T_r$ is single-valued;
(ii) for all $x, y \in E$, $\langle T_r x - T_r y, J_T r x - J_T r y \rangle \leq \langle x - y, J_T r x - J_T r y \rangle$;
(iii) $F(T_r) = JEP(f, C)$;
(iv) $JEP(f, C)$ is closed and convex.

Zeynab Jouymandi et al [44] introduced the following $J$-auxiliary equilibrium problem (shortly, $JAUEP$):

\[
\text{find } u \in C \text{ such that } \rho f(Ju, Jy) + L(Jy, Ju) \geq 0, \quad \forall y \in C.
\]

where $\rho > 0$ is a regularization parameter and $L : JC \times JC \to \mathbb{R}$ is a nonnegative differentiable convex bifunction on $JC$ with respect to the first argument, for any fixed $u \in C$, such that (i) $L(Ju, J\cdot) = 0$, $\nabla_1 L(Ju, J\cdot) = 0$ where $\nabla_1 L(Ju, J\cdot) = \text{gradient of } L(\cdot, Ju)$ at $Ju$.

Lemma 2.9 [44] Suppose that $C$ is a nonempty subset of a reflexive and smooth Banach space $E$, $JC$ is a nonempty and convex subset of $E^*$ and $f : JC \times JC \to \mathbb{R}$ is an equilibrium bifunction and let $u \in C$. Suppose that $f(Ju, \cdot) : JC \to \mathbb{R}$ is subdifferentiable and convex on $JC$. Let $L : JC \times JC \to R_+$ be a differentiable convex function on $JC$ with respect to first argument such that (i) $L(Ju, Ju) = 0$ and (ii) $\nabla_1 L(Ju, Ju) = 0$. Then, $u \in C$ is a solution to $JEP$ if $u$ is a solution to $JAUEP$.

Proposition 2.10 [45] Let $E$ be $p$-uniformly convex Banach space, $p \geq 1$, then the normal duality mapping $J_p$ of $E$ is the subdifferential of the function $\frac{1}{p} \| \cdot \|^p$.

Proposition 2.11 [46] Let $E$ be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of $E$. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.

3 Main results

In this section, we assume that bifunction $f : JC \times JC \to \mathbb{R}$ satisfies the following conditions, where $C$ is a nonempty convex and closed subset of 2-uniformly convex and 2-uniformly smooth Banach space $E$ with smoothness coefficient $k$.

(a) $f(x^*, x^*) = 0$ for all $x^* \in JC$;
(b) $f$ is strongly pseudomonotone on $JC$, i.e.

\[
f(x^*, y^*) \geq 0 \Rightarrow f(y^*, x^*) \leq -\gamma \phi_s(x^*, y^*), \forall x^*, y^* \in JC;
\]
(c) for all \( x^*, y^*, z^* \in JC \), \( \limsup_{t \to 0} f(tz^* + (1-t)x^*, y^*) \leq f(x^*, y^*) \);

(d) \( f(x^*, \cdot) \) is convex, lower semicontinuous and subdifferentiable on \( JC \) for every \( x^* \in JC \);

(e) \( f \) satisfies \( \phi_\ast \)-Lipschitz-type condition: \( \exists c_1, c_2 > 0 \), such that for every \( x^*, y^*, z^* \in JC \),

\[
f(x^*, y^*) + f(y^*, z^*) \geq f(x^*, z^*) - c_1 \phi_\ast(y^*, x^*) - c_2 \phi_\ast(z^*, y^*).
\]

(f) \( f \) is jointly weak* continuous on \( JC \times JC \), i.e., if \( \{ x^*_n \} \) and \( \{ y^*_n \} \) are two sequences in \( JC \) converging to \( x^* \in JC \) and \( y^* \in JC \) in weak* topology, respectively, then \( f(x^*_n, y^*_n) \to f(x^*, y^*) \).

It is known that if \( f \) satisfies the conditions (a)-(d), then \( JEP(f, C) \) is closed and convex.

**Theorem 3.1** Suppose that \( C \) is a nonempty closed convex subset of a 2-uniformly convex and 2-uniformly smooth Banach space \( E \). \( JC \) is a nonempty and convex subset of \( E^\ast \). Assume that \( f : JC \times JC \to R \) is a bifunction which satisfies the conditions (a)-(f) and \( JEP(f, C) \neq \emptyset \). \( c_1, c_2 \) are \( \phi_\ast \)-Lipschitz-type constants of \( f \), \( \gamma \) is strongly pseudomonotone constant of \( f \). Choose \( x_0, x_1 \in C \) and \( \{ \lambda_n \} \subset (0, 1], \{ \theta_n \} \subset [0, 1] \), define the following algorithm:

\[
\begin{align*}
\omega_n &= x_n + \theta_n(x_n - x_{n-1}), \\
Jx_{n+1} &= \arg\min_{y \in C} \{ \lambda_n f(J\omega_n, Jy) + \frac{1}{2} \phi_\ast(Jy, J\omega_n) \}.
\end{align*}
\]

If the following conditions hold:

1. \( \gamma > 2c_2, k \in (0, \frac{\sqrt{2}}{2}) \);
2. \( 0 < \lambda_n < \min\{ \frac{1}{2c_1}, \frac{1}{2c_2} \} \);
3. \( \{ \theta_n \} \) is non-decreasing and \( \theta_n \in [0, \theta_\ast] \) for some \( \theta_\ast \in [0, \frac{\sqrt{13-3}}{2}) \).

Then the sequence \( \{ x_n \} \) converges strongly to \( x^* \in JEP(f, C) \).

**Proof:** By the condition (d) and Lemma 2.4, 2.5 and 2.6, we have

\[
Jx_{n+1} = \arg\min_{y \in C} \{ \lambda_n f(J\omega_n, Jy) + \frac{1}{2} \phi_\ast(Jy, J\omega_n) \}
\]

\[
\iff 0 \in \lambda_n \partial_2 f(J\omega_n, Jx_{n+1}) + \frac{1}{2} \nabla_1 \phi_\ast(Jx_{n+1}, J\omega_n) + NJC(Jx_{n+1}).
\]

Using Proposition 2.10 and \( J^* = J^{-1} \), we derive that

\[
\frac{1}{2} \nabla_1 \phi_\ast(Jx_{n+1}, J\omega_n) = x_{n+1} - \omega_n.
\]

Then for \( \omega \in \partial_2 f(J\omega_n, Jx_{n+1}) \) and \( \varpi \in NJC(Jx_{n+1}) \), we have

\[
0 = \lambda_n \omega + x_{n+1} - \omega_n + \varpi.
\]

(3.2)
Therefore, from the definition of \( \partial_2 f(J\omega_n, Jx_{n+1}) \), we obtain for every \( y \in C \)
\[
\langle \omega, Jy - Jx_{n+1} \rangle \leq f(J\omega_n, Jy) - f(J\omega_n, Jx_{n+1}).
\]
Let \( y = y^* \in JEP(f, C) \) in the inequality above, we get
\[
\langle \omega, Jy^* - Jx_{n+1} \rangle \leq f(J\omega_n, Jy^*) - f(J\omega_n, Jx_{n+1}). \tag{3.3}
\]
Use the definition of \( NC(Jx_{n+1}) \), we know
\[
\langle \omega, Jx_{n+1} - Jy \rangle \leq 0. \tag{3.4}
\]
By equality (3.2), we have
\[
\lambda_n \langle \omega, Jx_{n+1} - Jy \rangle \leq \langle x_{n+1} - \omega_n, Jx_{n+1} - Jy \rangle.
\]
It follows from (3.4) that \( \lambda_n \langle \omega, Jx_{n+1} - Jy \rangle + \langle x_{n+1} - \omega_n, Jx_{n+1} - Jy \rangle \leq 0. \)
So we have
\[
\lambda_n \langle \omega, Jx_{n+1} - Jy \rangle \leq \langle x_{n+1} - \omega_n, Jy - Jx_{n+1} \rangle, \tag{3.5}
\]
for all \( y \in C \). Setting \( y = y^* \) in inequality (3.5), we get
\[
\langle x_{n+1} - \omega_n, Jy^* - Jx_{n+1} \rangle \geq \lambda_n \langle \omega, Jx_{n+1} - Jy^* \rangle.
\]
From (3.3), it is easy to see
\[
\langle x_{n+1} - \omega_n, Jy^* - Jx_{n+1} \rangle \geq \lambda_n \{ f(J\omega_n, Jx_{n+1}) - f(J\omega_n, Jy^*) \},
\]
i.e.
\[
\langle \omega - x_{n+1}, Jy^* - Jx_{n+1} \rangle \leq \lambda_n \{ f(J\omega_n, Jy^*) - f(J\omega_n, Jx_{n+1}) \}.
\]
Since \( f \) satisfies \( \phi_\star \)-Lipschitz-type condition, we have
\[
\langle \omega - x_{n+1}, Jy^* - Jx_{n+1} \rangle \leq \lambda_n \{ f(Jx_{n+1}, Jy^*) + c_1 \phi_\star(Jx_{n+1}, J\omega_n) + c_2 \phi_\star(Jy^*, Jx_{n+1}) \}. \tag{3.6}
\]
Because \( y^* \) is a solution of \( JEP \), then \( f(Jy^*, Jx_{n+1}) \geq 0. \) And owing to \( f \) is strongly pseudomonotone on \( JC \), we know \( f(Jx_{n+1}, Jy^*) \leq -\gamma \phi_\star(Jy^*, Jx_{n+1}) \).
Substituting this inequality into (3.6), we obtain
\[
\langle \omega - x_{n+1}, Jy^* - Jx_{n+1} \rangle \leq -\lambda_n \gamma \phi_\star(Jy^*, Jx_{n+1}) + \lambda_n c_1 \phi_\star(Jx_{n+1}, J\omega_n) + \lambda_n c_2 \phi_\star(Jy^*, Jx_{n+1}). \tag{3.7}
\]
Then we have
\[
2\langle \omega_n - x_{n+1}, Jy^* - Jx_{n+1} \rangle \leq -2\lambda_n \gamma \phi_*(Jy^*, Jx_{n+1}) + 2\lambda_n c_1 \phi_*(Jx_{n+1}, J\omega_n) + 2\lambda_n c_2 \phi_*(Jy^*, Jx_{n+1}).
\]

From equation (2.3) and the relationship between \( \phi \) and \( \phi_* \), we obtain that
\[
\phi(\omega_n, x_{n+1}) + \phi(x_{n+1}, y^*) - \phi(\omega_n, y^*) \leq -2\lambda_n \gamma \phi(x_{n+1}, y^*) + 2\lambda_n c_1 \phi(\omega_n, x_{n+1}) + 2\lambda_n c_2 \phi(x_{n+1}, y^*).
\]

Organizing the inequality above, we have
\[
(1 - 2\lambda_n c_2 + 2\lambda_n \gamma)\phi(x_{n+1}, y^*) \leq \phi(\omega_n, y^*) - (1 - 2\lambda_n c_1)\phi(\omega_n, x_{n+1}).
\]

It follows from the definition of \( \phi \) and Lemma 2.2 and Lemma 2.3 that
\[
\phi(\omega_n, y^*) = \|\omega_n\|^2 - 2\langle \omega_n, Jy^* \rangle + \|y^*\|^2 \\
= \|(1 + \theta_n)x_n - \theta_n x_{n-1}\|^2 - 2\langle \omega_n, Jy^* \rangle + \|y^*\|^2 \\
\leq (1 + \theta_n)\|x_n\|^2 - \theta_n\|x_{n-1}\|^2 + \theta_n(1 + \theta_n)\phi(x_{n-1}, x_n) \\
- 2(1 + \theta_n)\langle x_n, Jy^* \rangle + 2\theta_n\langle x_{n-1}, Jy^* \rangle + (1 + \theta_n)\|y^*\|^2 - \theta_n\|y^*\|^2 \\
\leq (1 + \theta_n)\phi(x_n, y^*) - \theta_n\phi(x_{n-1}, y^*) + \theta_n(1 + \theta_n)\phi(x_{n-1}, x_n),
\]

and from Lemma 2.1 and condition (1)
\[
-\phi(\omega_n, x_{n+1}) = -\|\omega_n\|^2 + 2\langle \omega_n, Jx_{n+1} \rangle - \|x_{n+1}\|^2 \\
= -\|x_n + \theta_n(x_n - x_{n-1})\|^2 + 2\langle x_n + \theta_n(x_n - x_{n-1}, Jx_{n+1} \rangle - \|x_{n+1}\|^2 \\
\leq -\|x_n\|^2 - 2\theta_n\langle x_n - x_{n-1}, Jx_n \rangle + 2\langle x_n, Jx_{n+1} \rangle \\
+ 2\theta_n\langle x_n - x_{n-1}, Jx_{n+1} \rangle - \|x_{n+1}\|^2 \\
\leq -\phi(x_n, x_{n+1}) + 2\theta_n\langle x_n - x_{n-1}, Jx_{n+1} - Jx_n \rangle \\
\leq -\phi(x_n, x_{n+1}) + 2\theta_n\|x_n - x_{n-1}\|^2 + \|Jx_n - Jx_{n+1}\|^2 \\
\leq -\phi(x_n, x_{n+1}) + \theta_n\|x_n - x_{n-1}\|^2 + \|x_n\|\|x_n - x_{n-1}\|^2 \\
+ \theta_n\|Jx_n\|^2 - 2\theta_n\langle Jx_{n+1}, x_n \rangle + \|Jx_{n+1}\|^2 \\
\leq -\phi(x_n, x_{n+1}) + \theta_n\phi(x_n, x_{n+1}) + \theta_n\phi(x_{n-1}, x_n) \\
\leq -(1 - \theta_n)\phi(x_n, x_{n+1}) + \theta_n\phi(x_{n-1}, x_n).
\]

Combining inequalities (3.8), (3.9) and (3.10), we get
\[
(1 - 2\lambda_n c_2 + 2\lambda_n \gamma)\phi(x_{n+1}, y^*) \leq (1 + \theta_n)\phi(x_n, y^*) - \theta_n\phi(x_{n-1}, y^*) \\
+ N_n\phi(x_{n-1}, x_n) - M_n\phi(x_n, x_{n+1}).
\]
where \( M_n = (1 - 2\lambda_n c_1)(1 - \theta_n) \), \( N_n = \theta_n(2 + \theta_n - 2\lambda_n c_1) \). Since \( \theta_* \in [0, \sqrt{\frac{3}{2}} - 3) \), we have

\[
0 \leq \frac{\theta_*(2 + \theta_*)}{1 - \theta_*} < 1.
\]

Let \( \sigma \) be fixed in \( (\frac{\theta_*(2 + \theta_*)}{1 - \theta_*}, 1) \), due to \( \lambda_n \to 0 \), then the existence of \( n_0 \geq 0 \) such that when \( n \geq n_0 \),

\[
1 - 2\lambda_n c_1 \geq \sigma. \tag{3.12}
\]

It follows from condition (1) and (3.11) that, for all \( n \geq n_0 \),

\[
\phi(x_{n+1}, y^*) \leq (1 - 2\lambda_n c_2 + 2\lambda_n \gamma)\phi(x_{n+1}, y^*) \\
\leq (1 + \theta_n)\phi(x_n, y^*) - \theta_n\phi(x_{n-1}, y^*) + N_n\phi(x_{n-1}, x_n) \\
- M_n\phi(x_n, x_{n+1}). \tag{3.13}
\]

Let \( \varphi_n = \phi(x_n, y^*) - \theta_n\phi(x_{n-1}, y^*) + N_n\phi(x_{n-1}, x_n) \), because \( \{\theta_n\} \) is non-decreasing and (3.13), then

\[
\varphi_{n+1} - \varphi_n = \phi(x_{n+1}, y^*) - \theta_{n+1}\phi(x_n, y^*) + N_{n+1}\phi(x_n, x_{n+1}) - \phi(x_n, y^*) \\
+ \theta_n\phi(x_{n-1}, y^*) - N_n\phi(x_{n-1}, x_n) \\
\leq \phi(x_{n+1}, y^*) - (1 + \theta_n)\phi(x_n, y^*) + N_{n+1}\phi(x_n, x_{n+1}) \\
+ \theta_n\phi(x_{n-1}, y^*) - N_n\phi(x_{n-1}, x_n) \\
\leq -M_n\phi(x_n, x_{n+1}) + N_{n+1}\phi(x_n, x_{n+1}) \\
\leq -(M_n - N_{n+1})\phi(x_n, x_{n+1}). \tag{3.14}
\]

By (3.12), condition (2) and \( \sigma \in (\frac{\theta_*(2 + \theta_*)}{1 - \theta_*}, 1) \), we obtain \( 0 < 1 - 2\lambda_{n+1} c_1 < 1 \), and

\[
M_n - N_{n+1} = (1 - 2\lambda_n c_1)(1 - \theta_n) - \theta_{n+1}(2 + \theta_{n+1} - 2\lambda_{n+1} c_1) \\
\geq (1 - \theta_n)\sigma - \theta_{n+1}(2 + \theta_{n+1}) \\
\geq (1 - \theta_{n+1})\sigma - \theta_{n+1}(2 + \theta_{n+1}) \\
\geq (1 - \theta_*)\sigma - \theta_*(2 + \theta_*) \\
= K > 0.
\]

From (3.14), we get

\[
\varphi_{n+1} - \varphi_n \leq -K\phi(x_n, x_{n+1}) \leq 0. \tag{3.15}
\]

This implies that \( \{\varphi_n\}_{n=n_0}^{+\infty} \) is non-increasing. From the definition of \( \varphi_n \), we have

\[
\varphi_n \geq \phi(x_n, y^*) - \theta_n\phi(x_{n-1}, y^*).
\]
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So for all $n \geq n_0$, we have $\phi(x_n, y^*) \leq \varphi_n + \theta_n \phi(x_{n-1}, y^*) \leq \varphi_{n_0} + \theta_* \phi(x_{n-1}, y^*)$. Thus, we get the follows through induction

$$\phi(x_n, y^*) \leq \varphi_{n_0}(1 + \theta_* + \cdots + \theta_*^{n-n_0}) + \theta_*^{n-n_0} \phi(x_{n_0}, y^*), \forall n \geq n_0.$$ 

Then we have

$$\phi(x_n, y^*) \leq \frac{\varphi_{n_0}}{1 - \theta_*} + \theta_*^{n-n_0} \phi(x_{n_0}, y^*). \quad (3.16)$$

Again from the definition of $\varphi_n$, we obtain $\varphi_{n+1} \geq -\theta_{n+1} \phi(x_n, y^*)$. So

$$-\varphi_{n+1} \leq \theta_{n+1} \phi(x_n, y^*) \leq \theta_* \phi(x_n, y^*) \leq \frac{\theta_* \varphi_{n_0}}{1 - \theta_*} + \theta_*^{n-n_0+1} \phi(x_{n_0}, y^*). \quad (3.17)$$

Hence, it follows from (3.15), that for all $N \geq n_0$

$$k \sum_{n=n_0}^{N} \phi(x_n, x_{n+1}) \leq \varphi_{n_0} - \varphi_{N+1} \leq \frac{\varphi_{n_0}}{1 - \theta_*} + \theta_*^{N-n_0+1} \phi(x_{n_0}, y^*).$$

As $N \to \infty$ in the inequality above and by condition (3), we obtained

$$\sum_{n=n_0}^{\infty} \phi(x_n, x_{n+1}) < +\infty, \quad (3.18)$$

this implies that

$$\lim_{n \to \infty} \phi(x_n, x_{n+1}) = 0. \quad (3.19)$$

From Lemma 2.11, we know

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \quad (3.20)$$

By (3.20), we get for all $m \in N$,

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_n\| \leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \|x_{n+m-2} - x_n\| \leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \|x_{n+m-2} - x_{n+m-3}\| + \cdots + \|x_{n+1} - x_n\| \to 0. \quad (3.21)$$

So the sequence $\{x_n\}$ is a Cauchy sequence, then the limit of $\{x_n\}$ exists and implies that the limit of $\{\omega_n\}$ exists. Let $x_n \to x^*$, now we prove that $x^* \in JEP(f, C)$. 

It follows from the definition of $Jx_{n+1}$ that
\[ \lambda_n f(Jx_{n+1}, Jx_n) + \frac{1}{2} \phi_*(Jx_{n+1}, Jx_n) \leq \lambda_n f(Jx_{n+1}, Jy) + \frac{1}{2} \phi_*(Jy, Jx_n), \]
(3.22)
for all $y \in C$. It follows from (3.1) that $\omega_n - x_n = \theta_n(x_n - x_{n-1})$, so from (3.20), we have $\lim_{n \to \infty} \|x_n - \omega_n\| = 0$. Further, we also have
\[ \lim_{n \to \infty} \|x_{n+1} - \omega_n\| = 0. \]
(3.23)
By letting $n \to \infty$ in inequality (3.22), it follows from (3.23), condition (a) and (f) and uniformly norm-to-norm continuity of $J$ that
\[ 0 \leq f(Jx^*, Jy) + \frac{1}{2} \phi_*(Jy, Jx^*), \]
for all $y \in C$, because of $0 < \lambda_n \leq 1$. Hence, letting $\frac{1}{2} \phi_*(Jy, Jx^*) = L(Jy, Jx^*)$, it follows from Lemma 2.9 that $x^* \in JEP(f, C)$. This proof os completed.

**Corollary 3.2** Suppose that $C$ is a nonempty closed convex subset of Hilbert space $E$. Assume that $f : C \times C \to R$ is a bifunction which satisfies the conditions (a)-(f) and $EP(f, C) \neq \emptyset$. $c_1, c_2$ are $\phi_*$-Lipschitz-type constants of $f$, $\gamma$ is strongly pseudomonotone constant of $f$. Choose $x_0, x_1 \in C$ and \{\lambda_n\} $\subseteq$ (0, 1], \{\theta_n\} $\subseteq$ [0, 1], define the following algorithm:
\[ \begin{align*}
\omega_n &= x_n + \theta_n(x_n - x_{n-1}), \\
x_{n+1} &= \text{argmin}_{y \in C} \{\lambda_n f(\omega_n, y) + \frac{1}{2} \|\omega_n - y\|^2\}. 
\end{align*} \]
(3.24)
If the following conditions hold:
1. $\gamma > 2c_2, k \in (0, \frac{\sqrt{2}}{2})$;
2. $0 \leq \lambda_n < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$;
3. \{\theta_n\} is non-decreasing and $\theta_n \in [0, \theta_*]$ for some $\theta_* \in [0, \frac{1}{3})$.
Then the sequence \{\lambda_n\} converges strongly to $x^* \in EP(f, C)$.

**Remark:** When $E$ is a real Hilbert space, Theorem 3.1 is reduced to Theorem 4.1 in [27].

**References**

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[34] Takahashi, W., Zembayashi, K., A Strong convergence theorem for the equilibrium problem with a bifunction defined on the dual space of a Banach space, *Proceeding of the 8th International Conference on Fixed


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