A Weak Method to Come Close to Solution of Goldbach Conjecture

Pingyuan Zhou
Beiyuan 35-210, Chengdu University of Technology
Sichuan 610059, P. R. China

This article is distributed under the Creative Commons by-nc-nd Attribution License.
Copyright © 2019 Hikari Ltd.

Abstract

In this note we proved that if \( L_n \geq P_n - n \log n \) for \( n \geq 1 \) then \( L_n \to \infty \) as \( n \to \infty \) using the prime number theorem and Rosser theorem, and Goldbach conjecture follows from the result, where \( L_n \) is the largest strong Goldbach number generated by the \( n \)-th prime \( P_n \) and denotes an even number such that every even number from 4 to \( L_n \) is the sum of two primes among the first \( n \) primes but \( L_n + 2 \) is not such a sum. It means \( P_n - n \log n \) is the smallest possible value of \( L_n \) for \( n \geq 1 \) to support Goldbach conjecture, therefore, Goldbach conjecture is true if it can be proven that \( L_n > P_n - n \log n \) for \( n \geq 1 \).

Mathematics Subject Classification: 11A41

Keywords: prime, largest strong Goldbach number, the prime number theorem, Rosser theorem, Goldbach conjecture

1. Introduction

Goldbach conjecture is one of the most famous unsolved problems in mathematics and the conjecture states that every even number greater than 2 is the sum of two primes. As is well known, many studies have been made to prove Goldbach conjecture but the conjecture remains unsolved for more than 270 years. We try to find a new way to come close to solution of Goldbach conjecture based on the existence of the largest value of consecutive even numbers greater than 2 to be the sum of two primes among the first \( n \) primes. In our previous works[1,2], from mathematical experiment data we discovered a fact that the \( n \)-th prime \( P_n \) will generate an even number \( L_n \) such that every even number from 4 to \( L_n \) is the sum
of two primes among the first \( n \) primes but \( L_n + 2 \) is not such a sum and some consecutive primes will generate the same value of \( L_n \) so that distribution of \( L_n \) is a step-shaped curve with continuous growths as Figure 3 in [2] shows. With the significant fact, \( L_n \) was defined as the largest strong Goldbach number generated by \( P_n[1] \) and it was conjectured that there are two limits such that

\[
\lim_{n \to \infty} \frac{L_n}{2P_n} = 1
\]

and

\[
\lim_{n \to \infty} \frac{L_n}{P_n + n \log n} = 1,
\]

which arises from found numerical evidence for the first 4000000 primes[2]. It is clear that if any of the two limits is proven then we will get a result that \( L_n \to \infty \) as \( n \to \infty \) to imply Goldbach conjecture, however, this is a very strong method. In order to get the same result by taking a weak method, in this note, we will prove that if \( L_n \geq P_n - n \log n \) for \( n \geq 1 \) then \( L_n \to \infty \) as \( n \to \infty \). It means there is a weak method to get the result that \( L_n \to \infty \) as \( n \to \infty \) and we say that the weak method will be supported by the prime number theorem and Rosser theorem.

### 2. Main Results

**Definition 2.1**[1] \( L_n \) is called the largest strong Goldbach number generated by the \( n \)-th prime \( P_n \) if \( L_n \) is an even number such that every even number from 4 to \( L_n \) is the sum of two primes among the first \( n \) primes but \( L_n + 2 \) is not such a sum.

**Remark 2.2** \( L_n \) is defined as the largest strong Goldbach number generated by \( P_n \) because we want to find the largest value of consecutive even numbers greater than 2 to be the sum of two primes among the first \( n \) primes but \( L_n \) is just such an even number.

**Lemma 2.3** Primes are consecutively increased and infinite.

*Proof.* It is obvious that \( P_n < P_{n+1} \) for \( n \geq 1 \), therefore, all primes are consecutively increased. It had been proven by different methods that there are infinitely many primes. Hence the lemma holds.

**Lemma 2.4** \( L_n > 0 \) for \( n \geq 1 \).

*Proof.* By Lemma 2.3 primes are consecutively increased. Thus the largest value of consecutive even numbers greater than 2 to be the sum of two primes among the first \( n \) primes must be contained by the largest value of consecutive even numbers greater than 2 to be the sum of two primes among the first \( n + 1 \) primes. It means known value of any \( L_n \) will be contained by every \( L_m \) for \( m > n \). In fact, it has been observed that some consecutive primes generate the same value of \( L_n[1,2] \). Therefore, we have \( L_m \leq L_{m+1} \) for \( n \geq 1 \). Since \( L_1 = 4 > 0 \) for \( n = 1 \), \( L_n > 0 \) for \( n \geq 1 \). Hence the lemma holds.
Lemma 2.5 The largest possible value of $L_n$ is $2P_n$ for $n \geq 1$.

Proof. By Definition 2.1 $L_n$ denotes the largest value of consecutive even numbers greater than 2 to be the sum of two primes among the first $n$ primes. Since $2P_n$ is the largest possible sum as Definition 2.1 describes (for example, $L_{29} = 218$ for $P_{29} = 109$ in [1]) and there is no an even number such that the even number is such a sum and is also greater than $2P_n = P_n + P_n$. Hence the largest possible value of $L_n$ is $2P_n$ for $n \geq 1$ and the lemma holds. \qed

Lemma 2.6 If $L_n$ is represented as $L_n = \alpha P_n$ for $n \geq 1$, then $\alpha > 0$ for $n \geq 1$ and the largest possible value of $\alpha$ is 2 for $n \geq 1$.

Proof. Since $L_n > 0$ for $n \geq 1$ by Lemma 2.4 and $P_n > 0$ for $n \geq 1$, $\alpha > 0$ for $n \geq 1$ if $L_n$ is represented as $L_n = \alpha P_n$ for $n \geq 1$. By Lemma 2.5 the largest possible value of $L_n$ is $2P_n$ for $n \geq 1$. Thus the largest possible value of $\alpha$ is 2 for $n \geq 1$ if $L_n$ is represented as $L_n = \alpha P_n$ for $n \geq 1$. Hence the lemma holds. \qed

Remark 2.7 Although both the largest possible value of $L_n$ and the largest possible value of $\alpha$ are easily proven as above statements do, it is more important to find the smallest possible value of $L_n$, or equivalently, the smallest possible value of $\alpha$ when $L_n$ being represented as $L_n = \alpha P_n$ for $n \geq 1$, and we will first study the smallest possible value of $\alpha$ as the starting point for solution of the problem.

Lemma 2.8 If $L_n$ is represented as $L_n = \alpha P_n$ for $n \geq 1$, then the smallest possible value of $\alpha$ is a function $\alpha = 1 - \frac{n \log n}{P_n}$ such that $\alpha = 1 - \frac{n \log n}{P_n} > 0$ for $n \geq 1$ and $\alpha = 1 - \frac{n \log n}{P_n}$ approaches 0 as $n$ goes to infinity.

Proof. When $L_n$ is represented as $L_n = \alpha P_n$ for $n \geq 1$, $\alpha > 0$ for $n \geq 1$ and the largest possible value of $\alpha$ is 2 for $n \geq 1$ by Lemma 2.6. Thus the smallest possible value of $\alpha$ should be a function such that value of the function is greater than 0 for $n \geq 1$ and the function approaches 0 as $n$ goes to infinity.

Let the smallest possible value of $\alpha$ be a function such that $\alpha = 1 - \frac{n \log n}{P_n}$.

Then $\alpha = 1 - \frac{n \log n}{P_n} > 0$ for $n \geq 1$ by Rosser theorem since the theorem states that $P_n > n \log n$ for $n \geq 1$ [3] so that $\frac{n \log n}{P_n} < 1$ for $n \geq 1$.

According to equivalent statement of the prime number theorem, $P_n \approx n \log n$ as $n$ grows without bound using asymptotic notation. It means the relative error of
this approximation approaches 0 as \( n \) grows without bound and there exists a limit such that
\[
\lim_{{n \to \infty}} \frac{n \log n}{P_n} = 1. \tag{2.1}
\]
Based on the limit, we get the result that \( \alpha = 1 - \frac{n \log n}{P_n} \) approaches 0 as \( n \) grows without bound. Hence the lemma holds. \( \square \)

**Remark 2.9** In order to come close to solution of Goldbach conjecture, we must consider the smallest possible value of \( \alpha \) under the condition as weak as possible and Lemma 2.8 reasonably presented such condition for finding an appropriate approach to Goldbach conjecture.

**Lemma 2.10** If the smallest possible value of \( \alpha \) is a function \( \alpha = 1 - \frac{n \log n}{P_n} \), then the smallest possible value of \( L_n \) is a function \( L_n = P_n - n \log n \).

**Proof.** If \( L_n \) is represented as \( L_n = \alpha P_n \) for \( n \geq 1 \), from Lemma 2.8 the smallest possible value of \( L_n \) can be represented as
\[
L_n = (1 - \frac{n \log n}{P_n})P_n = P_n - n \log n.
\]
Hence the lemma holds. \( \square \)

**Theorem 2.11** If \( L_n \geq P_n - n \log n \) for \( n \geq 1 \), then \( L_n \to \infty \) as \( n \to \infty \).

**Proof.** Suppose \( L_n \) is represented as \( L_n \) for \( n \geq 1 \). Let the largest possible value of \( \alpha \) be 2 as \( n \) goes to infinity by Lemma 2.6. Then we have
\[
L_n = 2P_n \text{ as } n \to \infty.
\]
Since \( P_n \to \infty \) as \( n \to \infty \) by Lemma 2.3, we get
\[
2P_n \to \infty \text{ as } n \to \infty.
\]
Thus we obtain
\[
L_n \to \infty \text{ as } n \to \infty. \tag{2.2}
\]
Let the smallest possible value of \( \alpha \) be a function \( \alpha = 1 - \frac{n \log n}{P_n} \). Then we have
\[
\alpha = 1 - \frac{n \log n}{P_n} > 0 \text{ for } n \geq 1 \text{ and } \alpha = 1 - \frac{n \log n}{P_n} \text{ approaches 0 as } n \text{ grows without bound by Lemma 2.8. From it we see the function } \alpha = 1 - \frac{n \log n}{P_n} \text{ satisfies the weakest condition about the smallest possible value of } \alpha. \text{ By Lemma 2.10, we get the smallest possible value of } L_n \text{ as follows}
A weak method to come close to solution of Goldbach conjecture

\[ L_n = P_n - n \log n. \]  \hspace{1cm} (2.3)

By the prime number theorem and asymptotic notation, the relative error arising from \( P_n \approx n \log n \) can be written as

\[ \frac{P_n - n \log n}{P_n} \]

to lead to the existence of a limit such that

\[ \lim_{{n \to \infty}} \frac{P_n - n \log n}{P_n} = 0, \]  \hspace{1cm} (2.4)

and it is obvious that (2.1) follows from (2.4).

Considering both (2.3) and (2.4), there are the following four cases which may lead to the existence of limit (2.4) since \( P_n \) approaches infinity as \( n \) grows without bound by Lemma 2.3. Case 1: \( L_n = P_n - n \log n \) is a constant as \( n \) grows without bound. Case 2: \( L_n = P_n - n \log n \) can be arbitrarily large as \( n \) grows without bound. Case 3: \( L_n = P_n - n \log n \) approaches 0 as \( n \) grows without bound. Case 4: \( L_n = P_n - n \log n \) approaches lower order infinity than \( P_n \) as \( n \) grows without bound. It is clear that the first three cases should be ruled out. Therefore, there is only Case 4 to be reasonable, that is, \( L_n = P_n - n \log n \) approaches lower order infinity than \( P_n \) as \( n \) grows without bound. The result means that if the smallest possible value of \( \alpha \) is a function \( \alpha = 1 - \frac{n \log n}{P_n} \) then we obtain

\[ L_n \rightarrow \infty \text{ as } n \rightarrow \infty. \]  \hspace{1cm} (2.5)

The results (2.2) and (2.5) mean \( L_n \rightarrow \infty \) as \( n \to \infty \) holds for \( 1 - \frac{n \log n}{P_n} \leq \alpha \leq 2 \) for \( n \geq 1 \) when \( L_n \) being represented as \( L_n = \alpha P_n \) for \( n \geq 1 \), or equivalently, \( L_n \rightarrow \infty \) as \( n \to \infty \) holds for \( P_n - n \log n \leq L_n \leq 2P_n \) for \( n \geq 1 \), that is, if \( L_n \geq P_n - n \log n \) for \( n \geq 1 \) then \( L_n \rightarrow \infty \) as \( n \to \infty \). Hence the theorem holds.

\[ \square \]

Corollary 2.12 (Goldbach conjecture) Every even number greater than 2 is the sum of two primes.

Proof. By Theorem 2.11 \( L_n \rightarrow \infty \) as \( n \rightarrow \infty \) if \( L_n \geq P_n - n \log n \) for \( n \geq 1 \). Considering \( L_n \) to be the largest strong Goldbach number generated by \( P_n \) and every even number from 4 to \( L_n \) to be the sum of two primes among the first \( n \) primes by Definition 2.1, we must get a result such that every even number from 4 to \( L_n \) is the sum of two primes among the first \( n \) primes as \( n \) grows without bound to accord with \( L_n \rightarrow \infty \) as \( n \rightarrow \infty \), which means that every even number greater than 2 is the sum of two primes. Hence Goldbach conjecture is true and the
corollary holds.

\[\square\]

**Remark 2.13** By Theorem 2.11 and Corollary 2.12, there is a weak method to come close to solution of Goldbach conjecture, that is, Goldbach conjecture is true if it can be proven that \( L_n > P_n - n \log n \) for \( n \geq 1 \).

### 3. Conclusion

Since \( L_n \) denotes the largest value of consecutive even numbers greater than 2 to be the sum of two primes among the first \( n \) primes, Goldbach conjecture will follow from a result such that \( L_n \to \infty \) as \( n \to \infty \). By introducing \( \alpha = 1 - \frac{n \log n}{P_n} \) for \( n \geq 1 \) to be the smallest possible value of \( \alpha \) when \( L_n \) being represented as \( L_n = \alpha P_n \) for \( n \geq 1 \), we proved that if \( L_n \geq P_n - n \log n \) for \( n \geq 1 \) then \( L_n \to \infty \) as \( n \to \infty \). The result means \( P_n - n \log n \) is the smallest possible value of \( L_n \) for \( n \geq 1 \) to support Goldbach conjecture, and the result has contained a condition with practical signification such that \( L_n > P_n - n \log n \) for \( n \geq 1 \), therefore, Goldbach conjecture is true if it can be proven that \( L_n > P_n - n \log n \) for \( n \geq 1 \) by finding mathematical representation of \( L_n \). The weak method presented in this note seems to be more acceptable than our previous proposed strong method by considering the existence of \( \lim_{n \to \infty} \frac{L_n}{2P_n} = 1 \) or \( \lim_{n \to \infty} \frac{L_n}{P_n + n \log n} = 1 \).

### References


Received: October 1, 2019; Published: October 24, 2019