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A Cosine Approximation to the Skew Normal Distribution

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Abstract

We propose a new approximation to the skew normal distribution, a cosine approximation ($CASN$). This distribution is in a closed form and easy to use. $CASN$ is especially useful in statistical inference as it approximates the tail probabilities with very small absolute errors. Graphical and numerical comparisons are conducted to compare the probability density functions of skew normal and the $CASN$.

Mathematics Subject Classification: 62E17

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1 Introduction

Following [2], the probability density function (pdf) of the skew–normal distribution for a random variable Z with parameter λ is given by

$$g(z) = 2\phi(z)\Phi(\lambda x), \quad -\infty < x < \infty, \quad -\infty < \lambda < \infty \tag{1}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal density and distribution function, respectively. Density function (1) can be expressed in the following form:

$$g(z) = 2\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\int_{-\infty}^{\lambda z}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt, \quad -\infty < z < \infty, \quad -\infty < \lambda < \infty \tag{2}$$

In section 2 we present a new approximate skew normal distribution and in section 3 we compare the approximation with the original skew normal density. We give the approximation with location and scale parameters in section 4 and in section 5, we present the conclusions.

2 A Cosine Approximation to Skew Normal Distribution

[5] suggested an approximation for the standard normal distribution using the *cosine* function over the interval $-\pi$ to π. Their approximation for the density function of the standard normal is given by:

$$f(x) = \frac{1}{2\pi}(1 + \cos(x)), \quad -\pi < x < \pi \tag{3}$$

The corresponding cumulative distribution function (CDF) is given by,

$$F(x) = \begin{cases} 0 & x < -\pi \\ \dfrac{\pi + x + \sin(x)}{2\pi} & -\pi \leq x < \pi \\ 1 & x \geq \pi \end{cases} \tag{4}$$

2.1 *pdf* of the New Cosine Approximation to the Skew–Normal

Now, using the cosine approximation to the normal distribution, we shall define an approximate pdf $h(x)$ to the skew normal distribution by replacing the cdf of the standard normal in (1) by the approximation in (4). Then the approximate function $h(x)$ to the pdf of the skew normal distribution will be:

$$h(x) = 2\phi(x)F(\lambda x), \quad -\infty < x < \infty, \quad -\infty < \lambda < \infty \tag{5}$$

where $F(\lambda x)$ is defined by (4) on three different intervals depending on the value of the skew factor λ. So, the approximate pdf of the skew normal $h(x)$ will be

$$h(x) = \begin{cases} 0 & x < -\pi/\lambda \\ 2\dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}\dfrac{\pi + \lambda x + \sin(\lambda x)}{2\pi} & -\pi/\lambda \leq x < \pi/\lambda \\ 2\dfrac{1}{\sqrt{2\pi}}e^{-x^2/2} & x \geq \pi/\lambda \end{cases} \tag{6}$$

As we can notice, $h(x)$ is easier to obtain for any value of the random variable X compared to the original skew normal density function in (2).

Following [1], let ϕ be a density function symmetric about 0 and F an absolutely continuous distribution function such that F' is symmetric about 0. Then $h(x) = 2\phi(x)F(\lambda x)$ is a proper density function for any real λ. It follows that $\int_{-\infty}^{\infty} 2\phi(x)F(\lambda x)dx = 1$, and hence $h(x)$ is a proper density.

2.2 *CDF* of the New Cosine Approximation to the Skew–Normal

Now consider the corresponding cumulative distribution of (6), $H(x)$; when $\lambda \neq 0$,

$$H(x) = \begin{cases} 0 & x < -\pi/\lambda \\ H_1(x) & -\pi/\lambda \leq x < \pi/\lambda \\ H_2(x) & x \geq \pi/\lambda \end{cases} \tag{7}$$

Where,

$H_1(x) = \int_{-\pi/\lambda}^{x} 2\dfrac{1}{\sqrt{2\pi}}e^{-z^2/2}\dfrac{\pi + \lambda z + \sin(\lambda z)}{2\pi}dz =$

$$\frac{\sqrt{2}}{2\pi^{3/2}}\left(\lambda\left(-\mathrm{e}^{-1/2\,x^2} + \mathrm{e}^{-1/2\,\frac{\pi^2}{\lambda^2}}\right) + \pi\int_{-\frac{\pi}{\lambda}}^{x}\mathrm{e}^{-1/2\,z^2}\,\mathrm{d}z + \int_{-\frac{\pi}{\lambda}}^{x}\mathrm{e}^{-1/2\,z^2}\sin(\lambda\,z)\,\mathrm{d}z\right)$$

$$H_2(x) = \int_{-\pi/\lambda}^{\pi/\lambda} 2\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\frac{\pi + \lambda x + \sin(\lambda x)}{2\pi}dx + \int_{\pi/\lambda}^{x} 2\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$$

Also it is clear that $H(-\infty) = 0$, $H(\infty) = 1$.

3 Comparison between the exact and the approximate skew normal distribution

In this section we compare the skew normal distribution with the cosine approximation (CASN). Figures 1, 2, 3, 4, 5, and 6 compare the skew normal distribution: $SN(\lambda)$ and the cosine approximation to the skew normal: $CASN(\lambda)$ for different values of the skew parameter λ. In these plots, the solid red lines represent the $SN(\lambda)$ and the dashed blue lines represent the $CASN(\lambda)$. As you may notice, approximation works really well for $\lambda < 1$. Maximum absolute error for $1 \leq \lambda \leq 5$ was high compared to other values of the skew parameter. For $\lambda > 5$ the approximation starts to result in lower maximum absolute errors.

It is interesting to notice that for any λ value, the *tail probabilities* of the skew normal and the approximation are very close. Therefore, this approximation can be used for inference including hypothesis testing for any given λ value as we are only interested in the *tail areas*. Table 1 shows the maximum absolute errors for different values of the skew parameter λ.

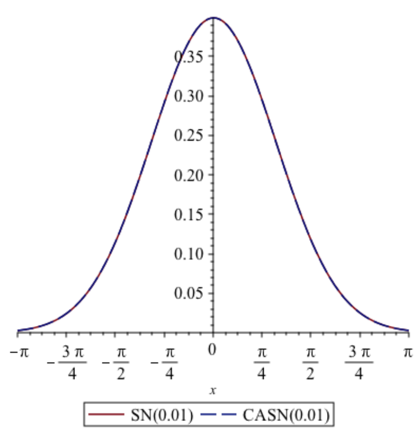

Figure 1:
The pdf for SN & CASN for $\lambda = 0.01$

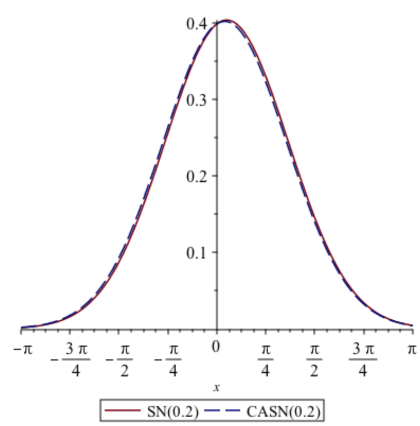

Figure 2:
The pdf for SN & CASN for $\lambda = 0.2$.

4 *CASN* with location ζ and scale ω

In this section we present the pdf, $l(x)$, of the $CASN$ with with location ζ and scale ω parameters.

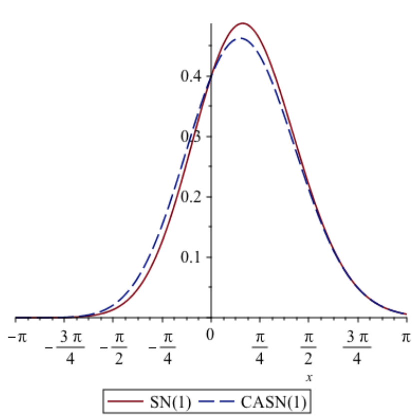

Figure 3:
The pdf for SN & CASN for $\lambda = 1$

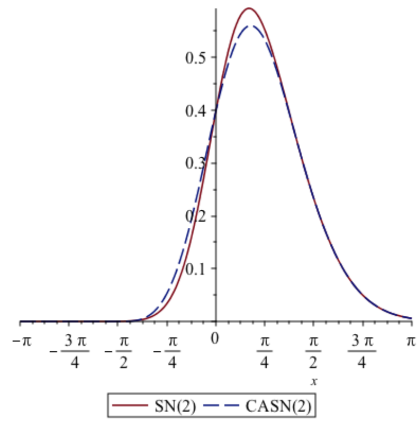

Figure 4:
The pdf for SN & CASN for $\lambda = 2$

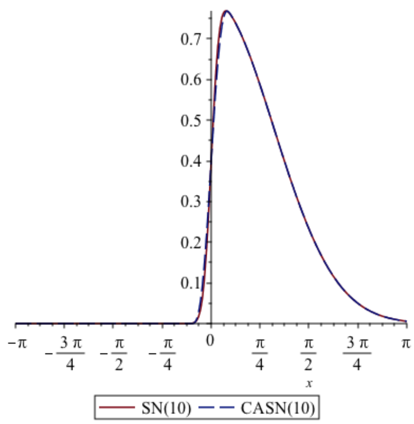

Figure 5:
The pdf for SN & CASN for $\lambda = 10$

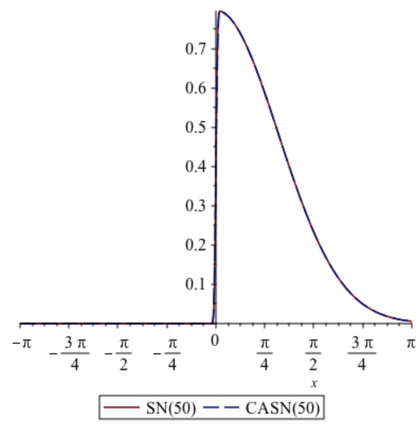

Figure 6:
The pdf for SN & CASN for $\lambda = 50$

λ	Maximum absolute error
0.01	0.00128
0.20	0.02460
1.00	0.06196
2.00	0.04623
10	0.01080
50	0.00217

Table 1: Maximum absolute error for CASN for a few λ values

$$l(x) = \begin{cases} 0 & x < -\pi/\lambda \\ 2\dfrac{1}{\sqrt{2\pi}}e^{-\frac{(x-\zeta)^2}{2\omega^2}}\dfrac{\pi + \lambda(\frac{x-\zeta}{\omega}) + \sin(\lambda\frac{x-\zeta}{\omega})}{2\pi} & -\pi/\lambda \leq x < \pi/\lambda \\ 2\dfrac{1}{\sqrt{2\pi}}e^{-\frac{(x-\zeta)^2}{2\omega^2}} & x \geq \pi/\lambda \end{cases} \tag{8}$$

Figure 7 shows an example of the pdf, $l(x)$ with location $\zeta = 0.2$, scale $\omega = 2$ and skew parameter $\lambda = 0.3$. The resulting maximum absolute error for this particular example was 0.03. Tables 2 and 3 give some simulation results for the absolute error when $\lambda = 0.3$ and $\lambda = 2$ for few different ζ and ω values.

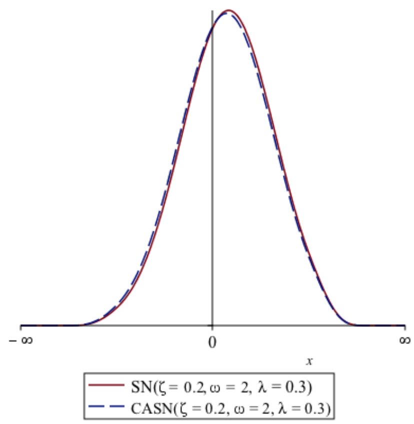

Figure 7:
The pdf for SN & CASN for $\zeta = 0.2$, $\omega = 2$ and $\lambda = 10$

ζ	ω	λ	Maximum absolute error
0.2	2	0.3	0.03540
1.0	2	0.3	0.03392
5.0	2	0.3	0.00142
0.2	5	0.3	0.01528
1.0	5	0.3	0.01277
5.0	5	0.3	0.09560

Table 2: Maximum absolute error for CASN for a $\lambda = 0.3$

4.1 Tail probabilities comparison

[4] was used to conduct a comparison of the tail probabilities. Tables 5 and 4 show the left tail and right tail probabilities of the Normal, SN and CASN distributions using the usual cutoffs for the standard normal distribution for different λ values. The Absolute Error column, presents the absolute difference between the SN and CASN tail probabilities. Results suggest that the tail probabilities (left or right) of the original skew normal distribution and the cosine approximation to the skew normal are very close.

ζ	ω	λ	Maximum absolute error
0.2	2	2	0.02498
1.0	2	2	0.07826
5.0	2	2	0.00219
0.2	5	2	0.01528
1.0	5	2	0.01253
5.0	5	2	0.10591

Table 3: Maximum absolute error for CASN for a $\lambda = 2$

λ	Cutoff	Normal	Skew Normal	CASN	Absolute Error
0.01	1.645	0.04998	0.05080	0.05064	0.00016
	1.960	0.02499	0.02546	0.02536	0.00009
	2.576	0.00499	0.00511	0.00508	0.00002
0.2	1.645	0.04998	0.06593	0.06290	0.00302
	1.960	0.02499	0.03397	0.03229	0.00167
	2.576	0.00499	0.00717	0.00678	0.00039
1	1.645	0.04998	0.09747	0.09600	0.00147
	1.960	0.02500	0.04937	0.04906	0.00031
	2.576	0.00500	0.00997	0.00998	0.00001
2	1.645	0.04998	0.09996	0.09997	0.00001
	1.960	0.02500	0.05000	0.05000	0.00000
	2.576	0.00500	0.01000	0.01000	0.00000
10	1.645	0.04998	0.09997	0.09997	0.00000
	1.960	0.02500	0.05000	0.05000	0.00000
	2.576	0.00500	0.01000	0.01000	0.00000
50	1.645	0.04998	0.09997	0.09997	0.00000
	1.960	0.02500	0.05000	0.05000	0.00000
	2.576	0.00500	0.01000	0.01000	0.00000

Table 4: Right Tail Probabilities

5 Conclusions

We presented a cosine approximation to the skew normal distribution. Graphical and numerical results show that the approximation is close to the original skew normal density and easier to calculate. [3] software can be use to easily calculate the probabilities using the proposed approximation. As the *CASN* approximate the tail probabilities well for any value of the skew parameter λ,

λ	Cutoff	Normal	Skew Normal	CASN	Maximum absolute error
0.01	-1.645	0.04998	0.04916	0.04933	0.00017
	-1.960	0.02500	0.02453	0.02463	0.00009
	-2.576	0.00500	0.00488	0.00491	0.00002
0.2	-1.645	0.04998	0.03403	0.03706	0.00303
	-1.960	0.02500	0.01602	0.01770	0.00168
	-2.576	0.00500	0.00282	0.00321	0.00039
1	-1.645	0.04998	0.00250	0.00397	0.00147
	-1.960	0.02500	0.00062	0.00094	0.00031
	-2.576	0.00500	0.00002	0.00001	0.00001
2	-1.645	0.04998	0.00001	0.00000	0.00001
	-1.960	0.02500	0.00000	0.00000	0.00000
	-2.576	0.00500	0.00000	0.00000	0.00000
10	-1.645	0.04998	0.00000	0.00000	0.00000
	-1.960	0.02500	0.00000	0.00000	0.00000
	-2.576	0.00500	0.00000	0.00000	0.00000
50	-1.645	0.04998	0.00000	0.00000	0.00000
	-1.960	0.02500	0.00000	0.00000	0.00000
	-2.576	0.00500	0.00000	0.00000	0.00000

Table 5: Left Tail Probabilities

it can be used for inference with a high level of accuracy.

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