ii – Open Sets in Topological Spaces

Amir A. Mohammed and Beyda S. Abdullah

Department of Mathematics
College of Education
University of Mosul, Mosul, Iraq

Copyright © 2019 Amir A. Mohammed and Beyda S. Abdullah. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we introduce a new class of open sets in a topological space called $ii$ – open sets. We study some properties and several characterizations of this class, also we explain the relation of $ii$ – open sets with many other classes of open sets. Furthermore, we define $iw$ – closed sets and $iiw$ – closed sets and we give some fundamental properties and relations between these classes and other classes such as $w$ – closed and $aw$ – closed sets.

Keywords: $\alpha$ – open set, $w$ – closed set, $i$ – open set

1. Introduction

Throughout this paper we introduce and study the concept of $ii$ – open sets in topological space $(X, \tau)$. The $ii$ – open set is defined as follows: A subset $A$ of a topological space $(X, \tau)$ is said to be $ii$ – open if there exist an open set $G$ in the topology $\tau$ of $X$, such that

i. $G \neq \emptyset$,
ii. $A$ is contained in the closure of $(A \cap G)$,
iii. interior points of $A$ equal $G$.

One of the classes of open sets that produce a topological space is $\alpha$ – open. This class of open sets has been introduced in 1965 [3]. In this paper we prove that the family of $ii$ – open sets are also produce a topological space (Theorem 2.10). The $ii$ – open set is a generalization of many classes of open sets we mention some of them namely semi – open sets, $\alpha$ – open sets and $w$ – open sets defined by Levine [2], Njastad [3] and Sundaram and John [5], respectively. Further, we study the relation of $ii$ – open sets with the following sets: $i$ – open set, $w$ – closed set and $aw$ – closed set introduced by Mohammed and Askander [1], Sundaram and
John [5] and Parimala et.al [4], respectively. Depending on \( i \) – open and \( ii \) – open sets we define \( iiw \) – closed and \( iiw \) – closed sets.

We present our work in two sections. In the first one, we define \( ii \) – open sets and we give many related examples, and we investigate the relationship with other classes of open sets. We prove some theorems to discuss their properties. In the second section, we discuss the relationship between \( iiw \)-closed sets with closed, \( w \) – closed, \( iw \) – closed and \( aw \) – closed sets. Further, we present some fundamental properties of \( iiw \) – closed sets with classes mentioned above.

2. \( ii \)-Open Sets In Topological Space

In this section, we introduce and study the notion of \( ii \) – open sets in a topological space and we obtain some of its basic properties. We recall the following definitions, which are useful in the sequel.

**Definition 2.1**
A subset \( A \) of a space \((X, \tau)\) is called
1. Semi – open set [2] if \( A \subseteq cl(int(A)) \).
2. \( \alpha \) – open set [3] if \( A \subseteq int(cl(int(A))) \).
3. \( i \) – open set [1] if there exist an open set \( G \in \tau(X) \) such that
   i. \( G \neq \emptyset, X \).
   ii. \( A \subseteq cl(A \cap G) \).

The complement of an \( i \) – open set is an \( i \) – closed set.
4. \( w \) – closed set [5] if \( cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi – open in \((X, \tau)\).
5. \( aw \) - closed set [4] if \( wcl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \alpha \) – open in \((X, \tau)\). The complement of an \( aw \) – closed set is an \( aw \) – open set.
6. \( o(X), so(X), ao(X), io(X), wc(X), awc(X) \) are family of open, semi – open, \( \alpha \) – open, \( i \) – open, \( w \) – closed, \( aw \) – closed sets respectively.

**Definition 2.2**
A subset \( A \) of a space \((X, \tau)\) is called \( int \) – open set if there exist an open set \( G \in o(X) \) and \( G \neq \emptyset, X \), such that \( int(A) = G \). The complement of the \( int \) – open set is called \( int \) – closed. We denote the family of all \( int \) – open sets of topological space by \( into(X) \) and the \( int \)-closed sets is denoted by \( intc(X) \).

**Example 2.3**
Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, X, \{b\}, \{b, c\}\} \). Here the closed sets in \((X, \tau)\) are
\[
\mathcal{C}(X, \tau) = \{\emptyset, X, \{a, c\}, \{a\}\}
\]
\[
into(X) = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}
\]
\[
intc(X) = \{\emptyset, X, \{a, c\}, \{c\}, \{a\}\}.
\]
Definition 2.4
A subset \( A \) of a space \((X, \tau)\) is called \( ii \) – open set if there exist an open set \( G \in o(X) \), such that
\[ \begin{align*}
  \text{i. } & G \neq \emptyset, X \\
  \text{ii. } & A \subseteq cl(A \cap G) \\
  \text{iii. } & int(A) = G.
\end{align*} \]
The complement of the \( ii \) – open set is called \( ii \) – closed set. We denote the family of all \( ii \) – open sets of topological space by \( ii o(X) \). This definition means that \( A \) is \( int \) – open and \( i \) – open set.

Example 2.5
Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, X, \{b\}, \{b, c\}\} \). Here
\[ 
  C(X, \tau) = \{\emptyset, X, \{a, c\}, \{a\}\} \\
  into(X) = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\} \\
  io(X) = \{\emptyset, X, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}\} \\
  ii o(X) = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\} 
\]

Remark 2.6
Note that for a topological space \((X, \tau)\) every \( ii \) – open set is \( i \) – open and \( int \) – open. Further, every semi – open is \( ii \) – open set. But the converse is not true as shown in the following example.

Example 2.7
Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, X, \{a\}, \{b, c\}\} \). Here
\[ 
  C(X, \tau) = \{\emptyset, X, \{b, c\}, \{a\}\} \\
  io(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{b, c\}\} \\
  into(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\} \\
  ii o(X) = \{\emptyset, X, \{a\}, \{b\}, \{b, c\}\} 
\]

Let \( A = \{b\} \). Then \( A \) is not \( ii \) – open, but \( A \) is an \( i \) – open set. Let \( B = \{a, b\} \). Then \( B \) is not \( ii \) – open. However \( B \) is \( int \) – open set.

Theorem 2.8
Every open set is \( ii \) – open set.

Proof. Let \( G \) be open set in \((X, \tau)\). Since \( G \subseteq cl(G \cap G) = cl(G) \), it follows that \( G \) is \( i \) – open. Also, \( G \) is \( int \) – open because \( int(G) = G \). Thus \( G \) is \( ii \) – open set.\( \blacksquare \)

The converse of the above theorem is not true in general as shown in the following example.

Example 2.9
Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, X, \{b\}, \{b, c\}\} \). Here
\[ 
  C(X, \tau) = \{\emptyset, X, \{a, c\}, \{a\}\} \\
  into(X) = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\} 
\]
\[io(X) = \{\emptyset, X, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}\}\]
\[iio(X) = \{\emptyset, X, \{b\}, \{b, c\}\}\]
Now, \(A = \{a, b\}\), is \(ii\) – open but \(A\) is not open.

**Theorem 2.10**

If \((X, \tau)\) is a topological space then \((X, ii o(x))\) is also topological space.

**Proof.** Let \(A\) be \(ii\) – open set and let \(x \in A\), we will prove the existence of open set, say \(G \in o(X)\) such that \(x \in G \subseteq A\). Since \(A\) is \(ii\) – open, it follows that there exist an open set \(G \in o(X)\) such that

i. \(G \neq \emptyset, X\)

ii. \(A \subseteq cl(A \cap G)\)

iii. \(int(A) = G\).

Therefore, \(x \in A\) implies that \(x \in cl(A)\) and \(x \in cl(G)\). If \(G\) is closed, then \(x \in G \subseteq A\) and \(A\) is open. If \(G\) is not closed and \(x \notin G\) then \(x \notin cl(G)\) because \(G\) is not closed. Since \(x \in A\) implies that \(x \in cl(A) \cap cl(G)\) implies that \(x \in cl(G)\). This implies a contradiction. Therefore \(A\) is open.

**Example 2.11**

Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, X, \{b\}, \{b, c\}\}\). Here
\[C(X, \tau) = \{\emptyset, X, \{a\}, \{a, c\}\}\]
\[iio(X) = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}\]
Note that \((X, \tau)\) is a topological space and \((X, ii o(x))\) is a topological space.

**Theorem 2.12**

Every \(\alpha\) – open set is \(ii\) – open.

**Proof.** Let \((X, \tau)\) be a topological space and \(A \subseteq X\) be \(\alpha\) – open set. Since \(A \subseteq int(cl(int(A))) \subseteq cl(int(A))\). Therefore \(A\) is semi – open. Since, there exist an open set, say, \(G \neq \emptyset, X\) satisfying \(int(A) \subseteq G\), it follows that \(int(A) \subseteq G \cap A\). Therefore \(A \subseteq cl(A \cap G)\). Thus, \(A\) is \(i\) – open. We shall prove that \(int(A) = G\).

Note that if \(int(A) \neq G\), for all \(G \in o(X)\), then \(cl(int(A)) \neq cl(G)\). From above inclusions we conclude that \(A \subseteq cl(int(A) \cap A \cap G)\). This implies that \(A \cap cl(G)\). That is a contradiction. Therefore, \(A\) is \(ii\) – open set.

The converse of the above theorem is not true in general as shown in the following example.

**Example 2.13**

Let \(X = \{a, b, c, d\}\) and \(\tau = \{\emptyset, X, \{a\}, \{b, c, d\}\}\).
\[C(X, \tau) = \{\emptyset, X, \{a\}, \{b, c, d\}\}\]
\[ao(X) = \{\emptyset, X, \{a\}, \{b, c, d\}\}\]
\[iio(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}\]
Now, \(A = \{c\}\) is \(ii\) – open but not \(\alpha\) – open.
Corollary 2.14
Every $\alpha$-open set is int-open.

Proof. Clear.■

The converse of the above corollary is not true in general as shown in the following example.

Example 2.15
Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Here
$C(X, \tau) = \{\emptyset, X, \{b, c\}, \{a\}\}$
$into(X) = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b\}, \{a, c\}\}$
$ao(X) = \{\emptyset, X, \{a\}, \{b, c\}\}$
Let $A = \{a, b\}$. Then $A$ is not $\alpha$-open. However $A$ is int-open set.

Remark 2.16
The following diagram shows that the relationships of $ii$-open sets with other sets.

3. $iw$ – Closed And $iiw$ – Closed Sets In Topological Space

We introduce and study the notion of $iw$ – closed and $iiw$ – closed sets in topological space and obtain some of its basic properties. The proof of main results in this section is similar to that in [4].

Definition 3.1
A subset $A$ of $(X, \tau)$ is called a $iw$ – closed set if $wcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $i$ – open in $(X, \tau)$. The complement of an $iw$ – closed set is $iw$ – open set.
Theorem 3.2
Every iw-closed set is w-closed set.
Proof. Let $A$ be an iw-closed set in $(X, \tau)$ and $U$ be any semi-open set in $X$ such that $A \subseteq U$. Since every semi-open set is $i$-open [1]. Since $A$ is closed, $\text{wcl}(A) \subseteq \text{cl}(A) \subseteq U$. This shows that $A$ is w-closed set in $(X, \tau)$.\]

The converse of the above theorem is not true in general as shown in the following example.

Example 3.3
Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$.
$C(X, \tau) = \{\emptyset, X, \{a\}, \{b, c\}\}$
$\text{wc}(X, \tau) = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$
let $A = \{a, b\}$. Then, $A$ is a w-closed set but it is not iw-closed set.

Theorem 3.4
Every iw-closed set is aw-closed set.
Proof. Let $A$ be an iw-closed set in $(X, \tau)$ and $U$ be any $\alpha$-open set in $X$ such that $A \subseteq U$. Since every $\alpha$-open set is $i$-open [1]. Since $A$ is closed, $\text{wcl}(A) \subseteq \text{cl}(A) \subseteq U$. This shows that $A$ is aw-closed set in $(X, \tau)$.\]

The converse of the above theorem is not true in general as shown in the following example.

Example 3.5
Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{b\}, \{b, c\}\}$.
$C(X, \tau) = \{\emptyset, X, \{a\}, \{b, c\}\}$
$\text{awc}(X, \tau) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$
let $A = \{a, b\}$. Then, $A$ is aw-closed but it is not iw-closed set.

Theorem 3.6
If $A$ is $i$-open and iw-closed set then $A$ is w-closed.
Proof. Let $U$ be semi-open such that $A \subseteq U$. We shall prove that $\text{cl}(A) \subseteq U$. Since $U$ is semi-open, this implies that $U$ is $i$-open. By assumption we have $\text{cl}(A) \subseteq U$. That is $A$ is w-closed.\]

Definition 3.7
A subset $A$ of $(X, \tau)$ is called $iiw$-closed set if $\text{wcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $ii$-open in $(X, \tau)$. The complement of an $iiw$-closed set is an $iiw$-open set. We denote the family of all $iiw$-closed sets of topological space by $\text{iiwc}(X)$.

Example 3.8
Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{b\}, \{b, c\}\}$.
$C(X, \tau) = \{\emptyset, X, \{a\}, \{a, c\}\}$, $\text{iiwc}(X, \tau) = \{\emptyset, X, \{a\}, \{a, c\}\}$
**Theorem 3.9**
Every $iiw$ – closed set is $aw$ – closed set.

*Proof.* Let $A$ be an $iiw$ – closed set in $(X, \tau)$ and $U$ be any $\alpha$ – open set in $X$ such that $A \subseteq U$. Since every $\alpha$ – open set is $ii$ – open. Since $A$ is closed, $\text{wcl}(A) \subseteq \text{cl}(A) \subseteq U$. This shows that $A$ is $aw$ – closed set in $(X, \tau)$.

The converse of the above theorem is not true in general as shown in the following example.

**Example 3.10**
Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{b\}, \{b, c\}\}$.

$C(X, \tau) = \emptyset, X, \{a\}, \{a, c\}\}$

$iiwc(X, \tau) = \emptyset, X, \{a\}, \{a, c\}\}$

$awc(X, \tau) = \emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$

Let $A = \{a, b\}$. Then $A$ is $aw$ – closed set but it is not $iiw$ – closed set.

**Theorem 3.11**
Every $iw$ – closed set is a $iiw$ – closed set.

*Proof.* Let $A$ be an $iw$ – closed set in $(X, \tau)$ and $U$ be any $ii$ – open set in $X$ such that $A \subseteq U$. Since every $ii$ – open set is $i$ – open. Since $A$ is closed, $\text{wcl}(A) \subseteq \text{cl}(A) \subseteq U$. This show that $A$ is $iiw$ – closed set in $(X, \tau)$.

**Theorem 3.12**
If $A$ is $ii$ – open and $iiw$ – closed set then $A$ is $w$ – closed.

*Proof.* Since $A \subseteq A$ and $A$ is $ii$ – open and $iiw$ – closed, we have $\text{wcl}(A) \subseteq A$. Thus $\text{wcl}(A) = A$. Hence $A$ is $w$ – closed set in $X$.

**Theorem 3.13**
Union of two $iiw$ – closed sets are $iiw$ – closed set.

*Proof.* Let $A$ and $B$ be two $iiw$ – closed sets. Let $G$ be any $ii$ – open set in $(X, \tau)$, such that $A \cup B \subseteq G$. Then $A \subseteq G$ and $B \subseteq G$. Since $A$ and $B$ are $iiw$ – closed set, $\text{wcl}(A) \subseteq G$ and $\text{wcl}(B) \subseteq G$. Therefore $\text{wcl}(A) \cup \text{wcl}(B) = \text{wcl}(A \cup B) \subseteq G$. Hence $A \cup B$ is $iiw$ – closed set.

**Remark 3.14**
The following diagram shows that the relationships of $iiw$ – closed sets with other known existing sets.
References


Received: January 23, 2019; Published: February 4, 2019