Generalization of Rodrigues’ Formula in Weyl Space

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Abstract
In this paper, a generalization of Rodrigues’s Formula in Weyl space is expressed.

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1 Introduction

A manifold with a conformal metric $g_{ij}$ and a symmetric connection $\nabla_k$ satisfying the compatibility condition

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \quad (1)$$

is called a Weyl space that will be denoted by $W_n(g_{ij}, T_k)$. The vector field $T_k$ is named the complementary vector field.

The prolonged derivative and prolonged covariant derivative of $A$ are, respectively defined by ([1,5])

$$\dot{\partial}_k A = \partial_k A - pT_k A \quad (2)$$

and

$$\dot{\nabla}_k A = \nabla_k A - pT_k A \quad (3)$$

where $A$ is a satellite of $g_{ij}$ with weight $\{p\}$. 
2 Generalization of Rodrigues’ Formula

Let us take a point on the line \( v \). Let us denote its coordinates by \( x^a + tv^a \) where \( v^a = t^i x^a_i + rn^a \) and \( g_{ab} v^a v^b = 1 \). Let \( x^a + tv^a \) describe a curve which is tangent to the line \( v \). Then the prolonged covariant derivative of \( x^a + tv^a \) in the direction of \( v^k \), which is tangent vector field of a curve \( C \) at a point on \( W_2 \), satisfies the following condition:

\[
v^a + v^k (\hat{\nabla}_k t) v^a + tv^k \hat{\nabla}_k v^a = m v^a \quad (a = 1, 2, 3; k = 1, 2)
\]

(4)

where \( m \) is to be determined.

Multiplying (4) by \( g_{ab} v^b \), we get

\[
g_{ab} v^a v^b + v^k \hat{\nabla}_k t + tg_{ab}(v^k \hat{\nabla}_k v^a)v^b = m \quad (b = 1, 2, 3)
\]

(5)
or

\[
g_{ab} v^a v^b + v^k \hat{\nabla}_k t = m
\]

(6)

where \( g_{ab} v^a v^b = 1 \) and \( (v^k \hat{\nabla}_k v^a)v^b = 0 \).

Using (6) in (4), we have

\[
v^a + v^k (\hat{\nabla}_k t) v^a + tv^k \hat{\nabla}_k v^a = (g_{ab} v^c v^b + v^k \hat{\nabla}_k t) v^a \quad (c = 1, 2, 3)
\]

(7)
or

\[
v^k (x^a_k + t \hat{\nabla}_k v^a - g_{ab} x^c_k v^b v^a) = 0
\]

(8)

where \( v^c = v^k x^c_k \) [7].

We know that:

1) \( g_{ab} v^a x^b_i = g_{ab}(t^i x^a_j + rn^a)x^b_i = g_{ij} t^i = t_i \quad (i, j = 1, 2) \)

(9)

2) \[
g_{ab}(\hat{\nabla}_k v^a)(\hat{\nabla}_l v^b) = g_{ab}(D^i_k x^a_i + D^a_k n^a)(D^b_l x^b_l + D^b_l n^b)
\]
\[
= g_{ij} D^i_k D^j_l + D^a_k D^a_l
\]
\[
= D^i_k D^j_l (g_{ij} + h_i h_l)
\]
\[
= G_{kl} \quad (l = 1, 2)
\]

(10)

where \( D_k = -h_i D^i_k \) [7].

3) \[
g_{ab}(\hat{\nabla}_k v^a)x^b_j = g_{ab}(D^i_k x^a_i + D^a_k n^a)x^b_j
\]
\[
= g_{ij} D^i_k
\]
\[
= D_{jk} \quad [4].
\]

(11)
Multiplying (8) by $g_{ad}(\hat{\nabla}_lv^d)$ ($d = 1, 2, 3$) and using (9), (10) and (11), we obtain

$$v^k_1(D_{kl} + tG_{kl}) = 0. \tag{12}$$

Eliminating the parameter $t$ in (12), we have

$$(D_{11}G_{12} - D_{12}G_{11})v^1_1v^1_1 + (D_{11}G_{22} - D_{22}G_{11})v^1_1v^2_1 + (D_{21}G_{22} - D_{22}G_{21})v^2_1v^2_1 = 0 \tag{13}$$

or implicitly

$$\varepsilon^{jl}D_{ij}G_{kl}v^i_1v^k_1 = 0 \tag{14}$$

or

$$e^{jl}D_{ij}G_{kl}v^i_1v^k_1 = 0 \tag{15}$$

where $\varepsilon^{jl} = e^{jl}/\sqrt{g}$, $e^{12} = 1$, $e^{21} = -1$, and $e^{11} = e^{22} = 0$.

The curves satisfying the equation (15) on $W_2$ constitute the net of curves. This net is named as the intersector net.

Using (10) and (11), we get from (15)

$$e^{jl}g_{hi}D^h_jD^m_k(g_{ms} + h_mh_s)v^i_1v^k_1 = 0 \quad (m, s = 1, 2) \tag{16}$$

or

$$(De^{hs}g_{hi}g_{ms}D^m_k - De^{hs}g_{hi}D_k^sg^qh^q)v^i_1v^k_1 = 0 \quad (q = 1, 2) \tag{17}$$

where $e^{jl}D^h_sD^i_l = De^{hs}$, $D = |D^h_j|$ and $-D_k = D^m_kh_m$ or

$$(Dge_{im}D^m_k - Dge_{iq}D_k^q)v^i_1v^k_1 = 0 \tag{18}$$

where $e^{hs}g_{hi}g_{ms} = ge_{im} \quad [7]$, $g = |g_{ij}|$.

Taking $m$ instead of $q$ in the second term of (18), we get

$$e_{im}(D^m_k - D_k^mh^m)v^i_1v^k_1 = 0 \tag{19}$$

where $D$ and $g$ are nonvanishing.

(19) is equivalent to (15).

On the other hand, the prolonged covariant derivative of $Y$: $v^a = t^ix_i^a + rn^a$ in the direction of $v^k$ is

$$v^k_1\hat{\nabla}_kv^a = v^k_1(\hat{\nabla}_kt^i - r\omega_{kl}g^{il})x_i^a + v^k_1(t^i\omega_{ik} + \hat{\nabla}_k r)n^a \tag{20}$$
\[
\psi^k \nabla_k \psi^a = \psi^k (\nabla_k t^i - r \omega_{kl} g^{di} - t^j \omega_{jk} \frac{t^i}{r} - \frac{1}{r} t^i \nabla_k r)x_i^a + \psi^k (\frac{t^i}{r} \omega_{ik} + \frac{1}{r} \nabla_k r) \psi^a
\]  
(21)

where \( n^a = \frac{1}{r} (t^a - t^i x_i^a) \).

Let us denote the tangential component of (21) by \(-DY\):

\[
DY = \psi^k (-\nabla_k t^i + r \omega_{kl} g^{di} + t^j \omega_{jk} \frac{t^i}{r} + \frac{1}{r} t^i \nabla_k r)x_i^a.
\]  
(22)

\( DY \) is tangent to \( C \) if and only if the following equation is satisfied for some scalar \( q \):

\[
\psi^k (\nabla_k t^i - \omega_{kl} g^{di} - t^j \omega_{jk} \frac{t^i}{r} - \frac{1}{r} t^i \nabla_k r)x_i^a = qx^a k
\]  
(23)

Multiplying (23) by \( g_{ab} x^b_m \), we have

\[
\psi^k (\nabla_k t^m - r \omega_{km} - t^j \omega_{jk} \frac{t^m}{r} - \frac{1}{r} t^m \nabla_k r) = q g_{km} \psi^k
\]  
(24)

where \( g_{ab} x^b_i x^a_m = g_{im}, \quad g_{im} g^{di} = \delta^i_m \) and \( g_{im} t^i = t_m \).

Multiplying (24) by \( g^{ms} \), we get

\[
g^{ms} \psi^k (\nabla_k t^m - r \omega_{km} - t^j \omega_{jk} \frac{t^m}{r} - \frac{1}{r} t^m \nabla_k r) = q \psi^s
\]  
(25)

where \( g_{km} g^{ms} = \delta^s_k \).

Multiplying (25) by \( \varepsilon_{sh} \psi^h \), we obtain

\[
\varepsilon_{sh} g^{ms} \psi^k (\nabla_k t^m - r \omega_{km} - t^j \omega_{jk} \frac{t^m}{r} - \frac{1}{r} t^m \nabla_k r) \psi^h = 0
\]  
(26)

where \( \varepsilon_{sh} \psi^s \psi^h = 0 \) (the directions coincide), or

\[
\varepsilon_{sh} \psi^k (\nabla_k t^s - r \omega_{km} g^{ms} - t^j \omega_{jk} \frac{t^s}{r} - \frac{1}{r} t^s \nabla_k r) \psi^h = 0
\]  
(27)

or

\[
\varepsilon_{sh} \psi^k (D_k^s - h^s D_k) \psi^h = 0
\]  
(28)

where \( D_k^s = \nabla_k t^s - r \omega_{km} g^{ms} \), \( D_k = t^j \omega_{jk} + \nabla_k r \), \( h^s = t^s / r \).

(26) is equivalent to (28). From here, we have seen that \( C \) is a curve of the intersector net.

Let \( \kappa \) denote the magnitude (signed magnitude) of \( DY \) in the direction of \( v \) which is the tangent vector field of the curve \( C \) of the intersector net. Then, we have

\[
\kappa v^i = \psi^k (-\nabla_k t^i + r \omega_{kl} g^{di} + t^j \omega_{jk} \frac{t^i}{r} + \frac{1}{r} t^i \nabla_k r).
\]  
(29)
Using (29) in (22), we have

\[ DY + \kappa v^i x^a_i = DY + \kappa v^a = 0 \quad (30) \]

(30) is a generalization of Rodrigues’s formula.

**Remark:** Generalization of Rodrigues’ formula in \( E^3 \) was obtained by Pan [6].

If \( Z : v^a = n^a \), then \( v^k \nabla_k v^a = v^k \nabla_k n^a \). Using (20) and (23), we have

\[ DZ = v^k \nabla_k n^a = -v^k \omega_{kl} g^{il} x^a_i \quad \text{and} \quad -v^k \omega_{kl} g^{il} x^a_i = q v^i x^a_i. \]

From here, we get \( v^k (w_{kl} - \kappa g_{kl}) = 0 \) where \( -\kappa = q \), \( \kappa = \omega_{ij} v^i v^j \) and \( \kappa \) is the normal curvature of \( \tilde{W}_2 \). This is the equation of lines of curvature [3] i.e. The net of lines of curvature is the intersector net of the normal congruence. Let \( \kappa \) denote the magnitude of \( DZ \) along a curve of the intersector net of \( Z \). Since \( v^k \omega_{kl} g^{il} = \kappa v^i \), we have \( DZ = \kappa v^a \) or \( DZ + \kappa v^a = 0 \) or \( v^k \nabla_k n^a + \kappa v^a = 0 \). This is a special case of (30).

**Theorem 2.1** If the tangential component of the prolonged covariant derivative of a congruence along a curve \( C \) is tangent to the curve \( C \) then \( C \) is a curve of the intersector net of the congruence.

**References**


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