On Symmetric Bi-Generalized Derivations of Incline Algebras

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Abstract

In this paper, we introduce the notion of symmetric bi-generalized derivation of incline algebras and investigated some related properties. Also, we introduce the notion of joinitive symmetric mapping and obtain some interesting results.

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1 Introduction

Z. Q. Cao, K. H. Kim and F. W. Roush [2] introduced the notion of incline algebras in their book. Some authors studied incline algebras and application. N. O. Alshehri [1] introduced the notion of derivation in incline algebras. In this paper, we introduce the concept of a symmetric bi-generalized derivation in incline algebras and give some properties of incline algebras. Also, In this paper, we introduce the notion of symmetric bi-generalized derivation of incline algebras and investigated some related properties. Also, we introduce the notion of joinitive symmetric mapping and obtain some interesting results.
2 Preliminary

An incline algebra is a set $K$ with two binary operations denoted by “+” and “∗” satisfying the following axioms, for all $x, y, z \in K$,

(K1) $x + y = y + x$,
(K2) $x + (y + z) = (x + y) + z$,
(K3) $x ∗ (y ∗ z) = (x ∗ y) ∗ z$,
(K4) $x ∗ (y + z) = (x ∗ y) + (x ∗ z)$,
(K5) $(y + z) ∗ x = (y ∗ x) + (z ∗ x)$,
(K6) $x + x = x$,
(K7) $x + (x ∗ y) = x$,
(K8) $y + (x ∗ y) = y$.

For convenience, we pronounce “+” (resp. “∗”) as addition (resp. multiplication). Every distributive lattice is an incline algebra. An incline algebra is a distributive lattice if and only if $x ∗ x = x$ for all $x \in K$. Note that $x \leq y \iff x + y = y$ for all $x, y \in K$. It is easy to see that “≤” is a partial order on $K$ and that for any $x, y \in K$, the element $x + y$ is the least upper bound of $\{x, y\}$. We say that ≤ is induced by operation +.

In an incline algebra $K$, the following properties hold, for all $x, y, a, b \in K$,

(K9) $x ∗ y \leq x$ and $y ∗ x \leq x$ for all $x, y \in K$,
(K10) $y \leq z$ implies $x ∗ y \leq x ∗ z$ and $y ∗ x \leq z ∗ x$, for all $x, y, z \in K$,
(K11) If $x \leq y$ and $a \leq b$, then $x + a \leq y + b$, and $x ∗ a \leq y ∗ b$.

Furthermore, an incline algebra $K$ is said to be commutative if $x ∗ y = y ∗ x$ for all $x, y \in K$. A map $f$ is isotone if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in K$.

A subincline of an incline algebra $K$ is a non-empty subset $M$ of $K$ which is closed under the addition and multiplication. A subincline $M$ is said to be an ideal if $x \in M$ and $y \leq x$ then $y \in M$. An element “0” in an incline algebra $K$ is a zero element if $x + 0 = x = 0 + x$ and $x ∗ 0 = 0 = 0 ∗ x$ for any $x \in K$. An non-zero element “1” is called a multiplicative identity if $x ∗ 1 = 1 ∗ x = x$ for any $x \in K$. A non-zero element $a \in K$ is said to be a left (resp. right) zero divisor if there exists a non-zero $b \in K$ such hat $a ∗ b = 0$ (resp. $b ∗ a = 0$) A
zero divisor is an element of $K$ which is both a left zero divisor and a right zero divisor. An incline algebra $K$ with multiplicative identity 1 and zero element 0 is called an integral incline if it has no zero divisors. By a homomorphism of inclines, we mean a mapping $f$ from an incline algebra $K$ into an incline algebra $L$ such that $f(x + y) = f(x) + f(y)$ and $f(x * y) = f(x) * f(y)$ for all $x, y \in K$. A map $f : K \rightarrow K$ is regular if $f(0) = 0$. A subincline $I$ of an incline algebra $K$ is said to be $k$-ideal if $x + y \in I$ and $y \in I$, then $x \in I$. Let $K$ be an incline algebra. An element $a \in K$ is called a additively cancellative if for all $a, b \in K$, $a + b = a + c \Rightarrow b = c$. If every element of $K$ is additively cancellative, it is called additively cancellative.

**Definition 2.1.** Let $K$ be an incline algebra. A mapping $D(\cdot, \cdot) : K \times K \rightarrow K$ is called symmetric if $D(x, y) = D(y, x)$ holds for all $x, y \in K$.

**Definition 2.2.** Let $K$ be an incline algebra and $x \in K$. A mapping $d(x) = D(x, x)$ is called trace of $D(\cdot, \cdot)$, where $D(\cdot, \cdot) : K \times K \rightarrow K$ is a symmetric mapping.

**Definition 2.3.** Let $K$ be an incline algebra and let $D : K \times K \rightarrow K$ be a symmetric mapping. We call $D$ a symmetric bi-derivation on $K$ if it satisfies the following condition

$$D(x * y, z) = D(x, z) * y + x * D(y, z)$$

for all $x, y, z \in K$.

**Lemma 2.4.** Let $K$ be an incline algebra and let $D : K \times K \rightarrow K$ be a symmetric bi-derivation of $K$. Then $D(0, x) = D(x, 0) = 0$ for all $x \in K$.

### 3 Symmetric bi-generalized derivations of incline algebras

In what follows, let $K$ denote an incline algebra with a zero element 0 unless otherwise specified.

**Definition 3.1.** Let $K$ be an incline algebra. A symmetric map $F : K \times K \rightarrow K$ is called a symmetric bi-generalized derivation of $K$ if there exists a symmetric bi-derivation $D$ such that

$$F(x * y, z) = F(x, z) * y + x * D(y, z)$$

for all $x, y, z \in K$.

**Example 3.2.** Let $K = \{0, a, b, 1\}$ be a set in which “+” and “∗” is defined by
Then it is easy to check that \((K, +, \ast)\) is an incline algebra. Define a map 
\[ D : K \times K \rightarrow K \]
by

\[
D(x, y) = \begin{cases} 
0 & \text{if } (x, y) = (0, 0), (0, a), (a, 0), (0, b), (b, 0), (0, 1), (1, 0) \\
a & \text{if } (x, y) = (a, a), (a, b), (b, a), (a, 1), (1, a) \\
b & \text{if } (x, y) = (b, b), (1, 1), (b, 1), (1, b) \\
1 & \text{if } (x, y) = (1, 1) 
\end{cases}
\]

Then it is easy to prove that \(D\) is a symmetric bi-derivation of \(K\). Also, define 
\[ F : K \rightarrow K \]
by

\[
F(x, y) = \begin{cases} 
0 & \text{if } (x, y) = (0, 0), (0, a), (a, 0), (0, b), (b, 0), (0, 1), (1, 0) \\
a & \text{if } (x, y) = (a, a), (a, b), (b, a), (a, 1), (1, a) \\
b & \text{if } (x, y) = (b, b), (1, 1), (b, 1), (1, b) \\
1 & \text{if } (x, y) = (1, 1) 
\end{cases}
\]

Then it is easily checked that \(F\) is a symmetric bi-generalized derivation associated with \(D\) of \(K\).

**Proposition 3.3.** Let \(D\) be a symmetric bi-derivation of \(K\). If \(F\) is a symmetric bi-generalized derivation associated with \(D\) of \(K\), then \(F(0, 0) = 0\).

**Proof.** Let \(F\) be a symmetric bi-generalized derivation associated with \(D\) of \(K\). Then we have

\[
F(0, 0) = F(0 \ast 0, 0) \\
= F(0, 0) \ast 0 + 0 \ast D(0, 0) \\
= 0 + 0 = 0
\]

\( \square \)

**Proposition 3.4.** Let \(D\) be a symmetric bi-derivation of \(K\). If \(F\) is a symmetric bi-generalized derivation associated with \(D\) of \(K\), then Then \(F(0, x) = F(x, 0) = 0\) for all \(x \in K\).

**Proof.** Let \(F\) be a symmetric bi-generalized derivation associated with \(D\) of \(K\). Then we have

\[
F(0, x) = F(0 \ast 0, x) \\
= F(0, x) \ast 0 + 0 \ast D(0, x) \\
= 0 + 0 = 0
\]
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for every $x \in K$. Similarly, $F(x, 0) = 0$ for every $x \in K$. □

**Proposition 3.5.** Let $D$ be a symmetric bi-derivation of $K$ and let $F$ be a symmetric bi-generalized derivation associated with $D$ of $K$. If $d$ is a trace of $F$, then $d$ is regular.

*Proof.* Let $d$ be a trace of $F$. Then
\[
d(0) = F(0, 0) = F(x \ast 0, 0)
\]
\[
= F(x, 0) \ast 0 + x \ast D(0, x)
\]
\[
= 0 + 0 = 0
\]
for every $x \in K$.

This completes the proof. □

**Proposition 3.6.** Let $D$ be a symmetric bi-derivation of $K$ and let $F$ be a symmetric bi-generalized derivation associated with $D$ of $K$. Then $F(x \ast y, z) \leq F(x, z) + D(y, z)$ for all $x, y, z \in K$.

*Proof.* Let $F$ be a symmetric bi-generalized derivation associated with $D$ of $K$. Then by (K9), we have $F(x, z) \ast y \leq F(x, z)$ and $x \ast D(y, z) \leq D(y, z)$ for all $x, y, z \in K$. Hence we obtain, by (K11)
\[
F(x \ast y, z) = F(x, z) \ast y + x \ast D(y, z)
\]
\[
\leq F(x, z) + D(y, z)
\]
for all $x, y, z \in K$. This completes the proof. □

**Proposition 3.7.** Let $D$ be a symmetric bi-derivation of integral incline $K$ and let $F$ be a symmetric bi-generalized derivation associated with $D$ of $K$. Then for all $x, y, z \in K$,

1. $a \ast F(x, y) = 0$ implies that $a = 0$ or $D = 0$.
2. $F(x, y) \ast a = 0$ implies that $a = 0$ or $D = 0$.

*Proof.* (1) Let $a \ast F(x, y) = 0$ for every $x, y \in K$. Replacing $x$ by $x \ast z$ in this relation, we get
\[
0 = a \ast F(x \ast z, y) = a \ast ((F(x, z) \ast y) + (x \ast D(y, z)))
\]
\[
= a \ast (F(x, z) \ast y) + a \ast (x \ast D(y, z))
\]
\[
= a \ast (x \ast D(y, z)).
\]
By putting $x = 1$, we have $a \ast D(y, z) = 0$ for all $y, z \in K$. Since $K$ is an integral incline, i.e., it has no zero divisors, $a = 0$ or $D(y, z) = 0$ for all $y, z \in K$. Hence $a = 0$ or $D = 0$.

(2) Similarly, we can prove (2). □
Let $K$ be an incline algebra and let $F$ be a symmetric bi-generalized derivation associated with symmetric bi-derivation $D$ of $K$. For a fixed element $a \in K$, let us define a map $d_a : K \to K$ such that $d_a(x) = F(x, a)$ for every $x \in K$.

**Proposition 3.8.** Let $D$ be a symmetric bi-derivation of $K$ and let $F$ be a symmetric bi-generalized derivation associated with $D$ of $K$. Then $d_a$ is regular.

**Proof.** Let $D$ be a symmetric bi-derivation of $K$ and let $F$ be a symmetric bi-generalized derivation associated with $D$ of $K$. Then

$$d_a(0) = F(0, a) = F(0 \ast 0, a) = F(0, a) \ast 0 + 0 \ast D(0, a) = 0 + 0 = 0$$

This completes the proof. \hfill $\square$

**Proposition 3.9.** Let $D$ be a symmetric bi-derivation of $K$ and let $F$ be a symmetric bi-generalized derivation associated with $D$ of $K$. Then $d_a(x \ast y) = d_a(x) \ast y + x \ast D(y, a)$ for all $x, y \in K$.

**Proof.** Let $D$ be a symmetric bi-derivation of $K$ and let $F$ be a symmetric bi-generalized derivation associated with $D$ of $K$. Then

$$d_a(x \ast y) = F(x \ast y, a) = F(x, a) \ast y + x \ast D(y, a) = d_a(x) \ast y + x \ast D(y, a)$$

for all $x, y \in K$. This completes the proof. \hfill $\square$

**Proposition 3.10.** Let $D$ be a symmetric bi-derivation of $K$ and let $F$ be a symmetric bi-generalized derivation associated with $D$ of $K$. Then $d_0(x) = 0$ for all $x \in K$.

**Proof.** Let $D$ be a symmetric bi-derivation of $K$ and let $F$ be a symmetric bi-generalized derivation associated with $D$ of $K$. Then

$$d_0(x) = F(x, 0) = F(0 \ast 0, x) = F(0, x) \ast 0 + 0 \ast D(0, x) = 0 + 0 = 0$$

for all $x \in K$. This completes the proof. \hfill $\square$

Let $K$ be an incline algebra and let $F : K \times K \to K$ be a symmetric mapping. We call $F$ a *joinitive mapping* if it satisfies

$$F(x + y, z) = F(x, z) + F(y, z)$$

for all $x, y, z \in K$. 
Proposition 3.11. Let $K$ be an incline algebra and let $F$ be a joinitive symmetric bi-generalized derivation associated with symmetric bi-derivation $D$ of $K$. Then $F(x \ast y, z) \leq F(x, z)$ for all $x, y, z \in K$.

Proof. Let $D$ be a symmetric bi-derivation of $K$ and let $F$ be a joinitive symmetric bi-generalized derivation associated with $D$ of $K$. Then

$$F(x, z) = F(x + x \ast y, z) = F(x, z) + F(x \ast y, z),$$

which implies that $F(x \ast y, z) \leq F(x, z)$ for all $x, y, z \in K$. This completes the proof. 

Proposition 3.12. Let $K$ be an incline algebra and let $d$ be a trace of joinitive symmetric bi-generalized derivation $F$ associated with symmetric bi-derivation $D$ of $K$. Then $x \ast D(x, y) \leq d(x)$ for all $x, y \in K$.

Proof. Let $d$ be a trace of joinitive symmetric bi-generalized derivation $F$ associated with $D$ of $K$. Then

$$d(x) = F(x, x) = F(x + x \ast y, x) = F(x, x) + F(x, x) \ast y + x \ast D(x, y) = d(x) + d(x) \ast y + x \ast D(x, y) = d(x) + x \ast D(x, y)$$

for all $x, y \in K$. This implies that $x \ast D(x, y) \leq d(x)$. This completes the proof. 

Proposition 3.13. Let $K$ be an incline algebra and let $F$ be a joinitive symmetric bi-generalized derivation associated with symmetric bi-derivation $D$ of $K$. Then $F$ is an isotone symmetric bi-generalized derivation of $K$.

Proof. Let $(x, y) \leq (z, t)$ for $x, y, z, t \in K$. Then we have $x + z = z$ and $y + t = t$, and so $(x, y) + (z, t) = (z, t)$. Hence we obtain

$$F(z, t) = F((x, y) + (z, t)) = F(x, y) + F(z, t)$$

for all $x, y, z, t \in K$. This implies that $F(x, y) \leq F(z, t)$ for all $x, y, z, t \in K$. This completes the proof.

Proposition 3.14. Let $K$ be an incline algebra and let $F$ be a joinitive symmetric bi-generalized derivation associated with symmetric bi-derivation $D$ of $K$. Then $d(x + y) = d(x) + d(y) + F(x, y)$ for all $x, y \in K$. 


Proof. Let $x, y \in K$.

\[
d(x + y) = F(x + y, x + y) = F(x, x + y) + F(y, x + y) \\
= F(x, x) + F(x, y) + F(y, x) + F(y, y) \\
= d(x) + d(y) + F(x, y).
\]

This completes the proof. \qed

**Proposition 3.15.** Let $D$ be a symmetric bi-derivation of $K$ and let $F$ be a joinitive symmetric bi-generalized derivation associated with $D$ of $K$. If $x \leq y$, then $F(x, z) \leq F(y, z)$ for all $z \in K$.

Proof. Let $x \leq y$. Then we have $x + y = y$ and

\[
F(y, z) = F(x + y, z) = F(x, z) + F(y, z),
\]

which implies that $F(x, z) \leq F(y, z)$ for all $z \in K$. This completes the proof. \qed

Let $F$ be a symmetric bi-generalized bi-derivation associated with symmetric bi-derivation $D$ of $K$. For fixed element $a \in K$, define a set $Fix_a(K)$ by

\[
Fix_a(K) = \{ x \in K \mid F(x, a) = x \}.
\]

**Proposition 3.16.** Let $D$ be a symmetric bi-derivation of $K$ and let $F$ be a joinitive symmetric bi-generalized derivation associated with $D$ of $K$. Then $Fix_a(K)$ is a subalgebra of $K$.

Proof. Let $x, y \in Fix_a(K)$. Then

\[
F(x + y, a) = F(x, a) + F(y, a) \\
= x + y,
\]

which implies that $x + y \in Fix_a(K)$. This completes the proof. \qed
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