Strong 2-Jordan Product Preserving Maps on Operator Algebras

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Abstract
Let \( R \) be a ring having unit 1 and an idempotent element \( e_1 \). Assume that \( f : R \to R \) is a surjective map. It is shown that, under some mild conditions, \( f \) satisfies \( \{f(a), f(e)\}_2 = \{a, e\}_2 \) for all \( a \in R \) and \( e \in \{e_1, 1-e_1, 1\} \) if and only if \( f(1) \) is in the center of \( R \) with \( f(1)^3 = 1 \) and \( f(a) = f(1)a \) holds for all \( a \in R \). As applications, such maps on prime rings, standard operator algebras and von Neumann algebras are characterized, respectively.

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1 Introduction
Let \( R \) be an associative ring and \( k \) a positive integer. Recall that the \( k \)-commutator of elements \( a, b \in R \) is defined by \( [a, b]_k = [[a, b]_{k-1}, b] \) with \( [a, b]_0 = a \) and \( [a, b]_1 = [a, b] = ab - ba \) (see [4]). Recall that a map \( f : R \to R \) is said to be strong \( k \)-commutativity preserving if \( [a, b]_k = [f(a), f(b)]_k \) for all \( a, b \in R \) ([7]). Obviously, strong \( k \)-commutativity preserving maps are the usual strong commutativity preserving maps if \( k = 1 \). For the case \( k = 1 \), see [5, 8] and the references therein. For the case \( k \geq 2 \), in [7], it is shown that a
satisfying maps on a unital prime ring with characteristic not 2 and containing
a nontrivial idempotent is strong 2-commutativity preserving if and only if it has the form \( a \mapsto \lambda a + h(a) \), where \( \lambda \) is an element in the extended centroid of the ring satisfying \( \lambda^3 = 1 \) and \( h \) is a map from the ring into its center. With \( k \) increasing, the problem of characterizing strong \( k \)-commutativity preserving maps becomes much more difficult. Let \( X \) be a complex Banach space with \( \dim X \geq 2 \) and \( \mathcal{A} \) be a standard operator algebra on \( X \). Hou and Qi in [1] proved that, if the range of a map \( f : \mathcal{A} \to \mathcal{A} \) contains all operators of rank \( \leq 1 \), then \( f \) is strong \( k \)-commutativity preserving if and only if there exist a functional \( h \) on \( \mathcal{A} \) and a complex scalar \( \lambda \) with \( \lambda^{k+1} = 1 \) such that \( f(A) = \lambda A + h(A)I \) for all \( A \in \mathcal{A} \).

On the other hand, \( \mathcal{R} \) is also a Jordan ring under Jordan product \( \{a, b\} = ab + ba \). Jordan product is a kind of important products and had been studied intensively by many authors. Motivated by the \( k \)-commutator, we can define the \( k \)-Jordan product of \( a, b \in \mathcal{R} \) by \( \{a, b\}_k = \{\{a, b\}_{k-1}, b\}_1 \), where \( \{a, b\}_0 = a \) and \( \{a, b\}_1 = \{a, b\} = ab + ba \). In addition, \( f \) is called to strong \( k \)-Jordan product preserving if \( \{f(a), f(b)\}_k = \{a, b\}_k \) for each \( a, b \in \mathcal{R} \) (see [10]).

Thus, a natural problem is how to characterize strong \( k \)-Jordan product preserving maps on rings or algebras. Assume that \( \mathcal{R} \) is a unital ring containing a nontrivial idempotent and \( f : \mathcal{R} \to \mathcal{R} \) is a surjective map. Wang and Qi [10] showed that, under some mild conditions, \( f \) is strong \( k \)-Jordan product preserving if and only if there exists a nontrivial idempotent \( e_1 \), then a surjective map \( f : \mathcal{A} \to \mathcal{A} \) satisfies \( \{f(a), f(e)\}_1 = \{a, e\}_1 \) for all \( a \in \mathcal{A} \) and \( e \in \{e_1, 1 - e_1\} \) if and only if \( f(1) \) is in the center of \( \mathcal{A} \), \( f(1)^2 = 1 \) and \( f(a) = f(1)a \) for all \( a \in \mathcal{A} \); in [12] gave a concrete form of strong 2-Jordan product preserving surjective maps on standard operator algebras, and particularly, showed that, if a surjective map \( \Phi : \mathcal{M} \to \mathcal{M} \) (here, \( \mathcal{M} \) is a properly infinite von Neumann algebra) satisfies \( \{\Phi(A), \Phi(P)\}_2 = \{A, P\}_2 \) for all \( A \in \mathcal{M} \) and all idempotents \( P \in \mathcal{M} \), then \( \Phi(A) = \Phi(I)A \) for all \( A \in \mathcal{M} \).

The purpose of this paper is to consider strong 2-Jordan product preserving surjective maps on general rings. Assume that \( \mathcal{R} \) is a unital ring with an idempotent element \( e_1 \) and \( f : \mathcal{R} \to \mathcal{R} \) is a surjective map. It is shown that, under some mild conditions, \( f \) satisfies \( \{f(a), f(e)\}_2 = \{a, e\}_2 \) for all \( a \in \mathcal{R} \) and \( e \in \{e_1, 1 - e_1, 1\} \) if and only if \( f(1) \) is in the center of \( \mathcal{R} \) with \( f(1)^3 = 1 \) and \( f(a) = f(1)a \) holds for all \( a \in \mathcal{R} \) (Theorem 2.1). As applications, such maps on prime rings, standard operator algebras and von Neumann algebras are characterized, respectively (Corollaries 2.2-2.4), which generalize the corresponding results in [10, 12].
2 Main result and its proof

In this section, we will give the main result in this paper and its proof.

**Theorem 2.1** Let $\mathcal{R}$ be a ring having unit 1 and an idempotent element $e_1$. Assume that the characteristic of $\mathcal{R}$ is not 2 and $f : \mathcal{R} \to \mathcal{R}$ is a surjective map. If $\mathcal{R}$ satisfies $a\mathcal{R}e_1 = \{0\} \Rightarrow a = 0$ and $a\mathcal{R}(1-e_1) = \{0\} \Rightarrow a = 0$, then

$$\{f(a), f(e)\}_2 = \{a, e\}_2$$

holds for all $a \in \mathcal{R}$ and $e \in \{e_1, 1-e_1, 1\}$ if and only if $f(1) \in \mathcal{Z}(\mathcal{R})$, the center of $\mathcal{R}$, $f(1)^3 = 1$ and $f(a) = f(1)a$ holds for all $a \in \mathcal{R}$.

For the convenience, write $e_1 = e$ and $e_2 = 1-e_1$. Then $\mathcal{R}$ can be written as $\mathcal{R} = \mathcal{R}_{11} + \mathcal{R}_{12} + \mathcal{R}_{21} + \mathcal{R}_{22}$, where $\mathcal{R}_{ij} = e_i \mathcal{R} e_j$ ($i, j \in \{1, 2\}$).

Now, we give the proof of Theorem 2.1.

**Proof of Theorem 2.1.** The “if” part is obvious. In the following, we will prove the “only if” part by checking several claims.

**Claim 1.** $f(0) = 0$.

By the surjectivity of $f$, there exists some $s \in \mathcal{R}$ such that $f(s) = -f(0)$. Letting $a = s$ and $e = 1$ in Eq.(1), we have

$$4s = \{s, 1\}_2 = \{f(s), f(1)\}_2$$

which means $\{s, 1\}_2 = 0$. Since the characteristic of $\mathcal{R}$ is not 2, one gets $s = 0$. That is, $f(0) = -f(0)$. It follows from $\text{char}\mathcal{R} \neq 2$ that $f(0) = 0$.

**Claim 2.** $f$ is additive, i.e. $f(a + b) = f(a) + f(b)$ holds for all $a, b \in \mathcal{R}$.

For any $a, b \in \mathcal{R}$, by the surjectivity of $f$, there exists some element $c \in \mathcal{R}$ such that $f(c) = f(a + b) - f(a) - f(b)$. Let $i \in \{1, 2\}$. Note that

$$ce_i + 2e_ic\bar{e}_i + e_i = \{c, e_i\}_2 = \{f(c), f(e_i)\}_2$$

which implies $e_i^2 = 0$. Thus $f(c) = 0$ by Claim 1, and so $f(a + b) = f(a) + f(b)$.

**Claim 3.** $f(e_i)^3 = e$, $f(e_i)e_j = e_j f(e_i)$ and $f(1)e_i = e_i f(1)$, $i, j \in \{1, 2\}$.

For $e \in \{1, e_1, e_2\}$, by taking $a = e$ in Eq.(1), we have $\{f(e), f(e)\}_2 = \{e, e\}_2$, which implies $f(e)^3 = e$. Thus, for $i \neq j \in \{1, 2\}$, one gets

$$f(e_i)e_i = f(e_i)f(e_i)^3 = f(e_i)^4 = f(e_i)^3 f(e_i) = e_i f(e_i),$$
and so
\[ f(e_i)e_j = f(e_i)(1 - e_i) = (1 - e_i)f(e_i) = e_jf(e_i). \]

Combining the above two equations and Claim 2 gives
\[ f(1)e_i = (f(e_i) + f(e_j))e_i = e_if(e_i) + e_if(e_j) = e_if(1), \quad 1 \leq i \neq j \leq 2. \]

Claim 4. \( f(e_i) \in R_{ii}, i \in \{1, 2\}. \)

By letting \( a = e_1 \) and \( e = e_2 \) in Eq.(1), one can obtain
\[ f(e_1)f(e_2)^2 + f(e_2)^2f(e_1) + 2f(e_2)f(e_1)f(e_2) = 0. \]

Multiplying by \( f(e_2)^2 \) from the left and the right in the above equation, respectively, by Claim 3, one gets
\[ f(e_2)^2f(e_1)f(e_2)^2 + e_2f(e_2)f(e_1) + 2e_2f(e_1)f(e_2) = 0 \tag{2} \]

and
\[ e_2f(e_1)f(e_2) + f(e_2)^2f(e_1)f(e_2)^2 + 2e_2f(e_2)f(e_1) = 0 \tag{3} \]

Comparing Eq.(2) and Eq.(3) yields
\[ e_2f(e_1)f(e_2) - e_2f(e_2)f(e_1) = 0. \tag{4} \]

Similarly, one can show
\[ e_1f(e_1)f(e_2) - e_1f(e_2)f(e_1) = 0. \tag{5} \]

Combining Eqs.(4)-(5) gives \( f(e_1)f(e_2) = f(e_2)f(e_1) \), and so
\[ f(1)f(e_i) = (f(e_i) + f(e_j))f(e_i) = f(e_i)(f(e_i) + f(e_j)) = f(e_i)f(1), \quad i \neq j \in \{1, 2\}. \]

On the other hand, taking \( a = e_i \) \((i = 1, 2)\) and \( e = 1 \) in Eq.(1) gives
\[ f(e_i)f(1)^2 + f(1)^2f(e_i) + 2f(1)f(e_i)f(1) = 4e_i. \]

It follows from \( \text{char}R \neq 2 \) that
\[ f(1)^2f(e_i) = e_i, \quad i = 1, 2. \]

Multiplying by \( f(1) \) from the left in the above equation, and by Claim 3, one achieves \( f(e_i) = f(1)e_i = e_if(e_i)e_i \in R_{ii}, i = 1, 2. \)

Claim 5. For any \( a \in R \), we have \( f(e_i)^2f(a)f(e_j) = f(e_i)f(a)f(e_j)^2, \)
\[ 1 \leq i \neq j \leq 2. \]

For any \( a \in R \), by Eq.(1), one has
\[ f(a)f(e_1)^2 + f(e_1)^2f(a) + 2f(e_1)f(a)f(e_1) = ae_1 + e_1a + 2e_1ae_1, \tag{6} \]
and similarly,
\[ f(a)f(e_2)^2 + f(e_2)^2 f(a) + 2f(e_2)f(a)f(e_2) = ae_2 + e_2a + 2e_2ae_2 \quad (7) \]

and
\[ f(a)f(1)^2 + f(1)^2 f(a) + 2f(1)f(a)f(1) = 4a. \quad (8) \]

As \( e_1 + e_2 = 1 \), by Claims 2 and 4, Eq.(8) can be rewritten as
\[
\begin{align*}
&f(a)f(e_1)^2 + f(a)f(e_2)^2 + f(e_1)^2 f(a) + f(e_2)^2 f(a) + 2f(e_1)f(a)f(e_1) \\
&+ 2f(e_1)f(a)f(e_2) + 2f(e_2)f(a)f(e_1) + 2f(e_2)f(a)f(e_2) = 4a.
\end{align*}
\]

Combining Eq.(9) and Eqs.(2)-(3), and noting that \( \text{char} \mathcal{R} \neq 2 \), one obtains
\[
f(e_1)f(a)f(e_2) + f(e_2)f(a)f(e_1) = e_1ae_2 + e_2ae_1.
\]

This implies \( f(e_1)f(a)f(e_2) = e_1ae_2 \) and \( f(e_2)f(a)f(e_1) = e_2ae_1 \) by Claim 4.

For any \( a \in \mathcal{R} \), note that Eq.(8) holds. Multiplying by \( f(1) \) from the left and the right in Eq.(6), respectively, and by Claim 4, one gets
\[
e_2f(a)f(e_1)^2 = e_2ae_1 \quad \text{and} \quad f(e_1)^2 f(a)e_2 = e_1ae_2.
\]

So
\[
f(e_1)^2 f(a)e_2 = f(e_1)f(a)f(e_2) \quad \text{and} \quad e_2f(a)f(e_1)^2 = f(e_2)f(a)f(e_1).
\]

It follows from Claim 4 that \( f(e_1)^2 f(a)f(e_2) = f(e_1)f(a)f(e_2)^2 \) and \( f(e_2)f(a)f(e_1)^2 = f(e_2)^2 f(a)f(e_1) \).

Claim 6. \( f(1) \in \mathcal{Z}(\mathcal{R}) \).

For any \( a \in \mathcal{R} \), note that Eq.(8) holds. Multiplying by \( f(1) \) from the left and the right in Eq.(8), respectively, by Claim 3, one gets
\[
f(1)f(a)f(1)^2 + f(a) + 2f(1)^2 f(a)f(1) = 4f(1)a
\]

and
\[
f(a) + f(1)^2 f(a)f(1) + 2f(1)f(a)f(1)^2 = 4af(1).
\]

Comparing the above two equations gives
\[
4[af(1) - f(1)a] = f(1)f(a)f(1)^2 - f(1)^2 f(a)f(1). \quad (10)
\]

Note that, by Claims 2-5, we have
\[
e_1[f(1)f(a)f(1)^2 - f(1)^2 f(a)f(1)]e_2 = e_1[f(e_1)f(a)f(e_1)^2 + f(e_1)f(a)f(e_2)^2 + f(e_2)f(a)f(e_1)^2 + f(e_2)f(a)f(e_2)^2 \\
- 2f(e_1)^2 f(a)f(e_1) - 2f(e_1)^2 f(a)f(e_2) - 2f(e_2)^2 f(a)f(e_1) - 2f(e_2)^2 f(a)f(e_2)]e_2 = 0;
\]

and similarly, \( e_2[f(1)f(a)f(1)^2 - f(1)^2 f(a)f(1)]e_1 = 0 \). These and Eq.(10) imply
\[
e_1[af(1) - f(1)a]e_2 = e_2[af(1) - f(1)a]e_1 = 0.
\]
Now, by Claim 3, it is clear that
\[ f(1) = e_1f(1)e_1 + e_2f(1)e_2. \]
By a direct calculation, one can obtain
\[ e_1[af(1) - f(1)a]e_2 = e_1ae_2f(1)e_2 - e_1f(1)e_1ae_2 = 0 \]
and
\[ e_2[af(1) - f(1)a]e_1 = e_2ae_1f(1)e_1 - e_2f(1)e_2ae_1 = 0 \]
for all \( a \in \mathcal{R} \). That is,
\[ e_1ae_2f(1)e_2 = e_1f(1)e_1ae_2 \text{ and } e_2ae_1f(1)e_1 = e_2f(1)e_2ae_1 \]
hold for all \( a \in \mathcal{R} \).

Note that, by [9, Lemma 3.1], the center of \( \mathcal{R} \) is
\[ Z(\mathcal{R}) = \{ z_{11} + z_{22} : z_{11} \in \mathcal{R}_{11}, z_{22} \in \mathcal{R}_{22}, z_{11}a_{12} = a_{12}z_{22} \text{ and } z_{22}a_{21} = a_{21}z_{11} \text{ for all } a_{12} \in \mathcal{R}_{12}, a_{21} \in \mathcal{R}_{21} \}. \]
It follows that \( f(1) = e_1f(1)e_1 + e_2f(1)e_2 \in Z(\mathcal{R}) \).

Claim 7. \( f(a) = f(1)a \) holds for all \( a \in \mathcal{R} \).

Letting \( e = 1 \) in Eq.(1), and by Claim 6, one gets
\[ 4a = \{ a, 1 \}_2 = \{ f(a), f(1) \}_2 = 4f(1)^2f(a) \text{ for all } a \in \mathcal{R}, \]
which implies \( f(1)^2f(a) = a \) as \( \text{char} \mathcal{R} \neq 2 \). Multiplying \( f(1) \) from the left in the equation and noting that \( f(1)^3 = 1 \), we achieve \( f(a) = f(1)a \) for all \( a \in \mathcal{R} \).

The proof of Theorem 2.1 is finished.

Recall that a ring \( \mathcal{R} \) is called prime if, for any \( a, b \in \mathcal{R} \), \( a\mathcal{R}b = \{0\} \Rightarrow a = 0 \) or \( b = 0 \). It is obvious that prime rings satisfy the assumption “\( a\mathcal{R}e_1 = \{0\} \Rightarrow a = 0 \) and \( a\mathcal{R}(1 - e_1) = \{0\} \Rightarrow a = 0' \) in Theorem 2.1.

Applying Theorem 2.1 to prime rings, we have

Corollary 2.2 Let \( \mathcal{R} \) be a unital prime ring with an idempotent element \( e_1 \). Assume that the characteristic of \( \mathcal{R} \) is not 2 and \( f : \mathcal{R} \rightarrow \mathcal{R} \) is a surjective map. Then \( f \) satisfies \( \{ f(a), f(e) \}_2 = \{ a, e \}_2 \) for all \( a \in \mathcal{R} \) and \( e \in \{ e_1, 1 - e_1, 1 \} \) if and only if \( f(1) \) is in the center of \( \mathcal{R} \) with \( f(1)^3 = 1 \) and \( f(a) = f(1)a \) holds for all \( a \in \mathcal{R} \).

Let \( X \) be a Banach space with dimension greater than 1. Denote by \( \mathcal{B}(X) \) the algebra of all bounded linear operators on \( X \). Recall that a standard operator algebra on \( X \) is a subalgebra of \( \mathcal{B}(X) \) which contains the identity operator and all finite-rank operators in \( \mathcal{B}(X) \). It is well known that standard operator algebras are prime. Hence, by Corollary 2.2, the following result is obvious, which generalizes Theorem 2.1 in [12].
Corollary 2.3 Let $X$ be a Banach space with $\dim X > 1$ and $A$ a standard operator algebra on $X$. Assume that $\Phi : A \to A$ is a surjective map. Then $f$ satisfies $\{\Phi(A), \Phi(P)\}_2 = \{A, P\}_2$ for all $A \in A$ and all idempotents $P \in A$ if and only if $\Phi(A) = \lambda A$ holds for all $A \in A$, where $\lambda$ is a scalar with $\lambda^3 = 1$.

Recall that a von Neumann algebra $\mathcal{M}$ is a subalgebra of some $\mathcal{B}(H)$, the algebra of all bounded linear operators acting on a complex Hilbert space $H$, which satisfies the double commutant property: $\mathcal{M}'' = \mathcal{M}$, where $\mathcal{M}' = \{T : T \in \mathcal{B}(H) \text{ and } TA = AT \forall A \in \mathcal{M}\}$ and $\mathcal{M}'' = \{\mathcal{M}'\}'$ ([2], [3]). For $A \in \mathcal{M}$, the central carrier of $A$, denoted by $\overline{A}$, is the intersection of all central projections $P$ such that $PA = 0$. If $A$ is self-adjoint, then the core of $A$, denoted by $\mathcal{C}(A)$, is the intersection of all central projections $P$ such that $PA = 0$. A projection $P$ is core-free if $P = 0$. It is easy to see that $P = 0$ if and only if $I - P = I$. If $\mathcal{M}$ has no central summands of type $I_1$, then by [6], each nonzero central projection $C \in \mathcal{M}$ is the carrier of a core-free projection in $\mathcal{M}$; particularly, there exists a nonzero core-free projection $P \in \mathcal{M}$ with $P = I$. For such $P$, note that $P = I - P = I$. It follows from the definition of the central carrier that both span\{$TP(x) : T \in \mathcal{M}, x \in H$\} and span\{$T(I - P)(x) : T \in \mathcal{M}, x \in H$\} are dense in $H$. So $\mathcal{AMP} = \{0\}$ implies $A = 0$ and $\mathcal{AM}(I - P) = \{0\}$ implies $A = 0$. Thus, if $\mathcal{M}$ has no central summands of type $I_1$, then $\mathcal{M}$ satisfies the corresponding assumption in Theorem 2.1. Therefore, By Theorem 2.1, the following result is true.

Corollary 2.4 Let $\mathcal{M}$ be a von Neumann algebra without central summands of type $I_1$. Then a surjective map $\Phi : \mathcal{M} \to \mathcal{M}$ satisfies $\{\Phi(A), \Phi(P)\}_2 = \{A, P\}_2$ for all $A \in \mathcal{M}$ and all idempotent operators $P \in \mathcal{M}$ if and only if $\Phi(I) \in \mathcal{Z}(\mathcal{M})$ with $\Phi(I)^3 = I$ and $\Phi(A) = \Phi(I)A$ for all $A \in \mathcal{M}$.

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References


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