The Quotient $\frac{\Omega(n)}{\omega(n)}$

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Abstract

We study the sum

$$\sum_{n=1}^{N} \left( \frac{\Omega(n)}{\omega(n)} \right)^{\alpha}$$

where $\alpha$ is an arbitrary but fixed real number. In our main theorem we prove the limit

$$\lim_{N \to \infty} \frac{\sum_{n=1}^{N} \left( \frac{\Omega(n)}{\omega(n)} \right)^{\alpha}}{N} = 1$$

As usual, $\omega(n)$ denotes the number of distinct primes in the prime factorization of $n$ and $\Omega(n)$ denotes the total number of primes in the prime factorization of $n$.

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1 Introduction and Main Results

In this article (see either [1] or [2, Chapter XXII]) $\omega(n)$ denotes the number of distinct primes in the prime factorization of $n$ and $\Omega(n)$ denotes the total number of primes in the prime factorization of $n$. That is, if the prime factorization of $n$ is $q_1^{a_1} \cdots q_r^{a_r}$, where the $q_i$ ($i = 1, \ldots, r$) ($r \geq 1$) are the different
primes and the \( s_i \) \((i = 1, \ldots, r)\) are the multiplicities or exponents, we have \( \omega(n) = r \) and \( \Omega(n) = s_1 + \cdots + s_r \).

The functions \( \omega(n) \) and \( \Omega(n) \) were studied by G. H. Hardy and S. Ramanujan in 1917 [1]. They obtained, among other results, the following formulae

\[
\sum_{n \leq x} \omega(n) = x \log \log x + Mx + o(x),
\]

\[
\sum_{n \leq x} \Omega(n) = x \log \log x + \left( M + \sum_p \frac{1}{p(p-1)} \right) x + o(x),
\]

where \( M \) is Mertens’s constant. In the same paper they define the normal order of an arithmetical function and they prove that the normal order of \( \omega(n) \) and \( \Omega(n) \) is \( \log \log n \).

In this article we study the sum

\[
\sum_{n=1}^{N} \left( \frac{\Omega(n)}{\omega(n)} \right)^{\alpha}
\]

where \( \alpha \) is an arbitrary but fixed real number. In our main theorem we shall prove the limit

\[
\lim_{N \to \infty} \frac{\sum_{n=1}^{N} \left( \frac{\Omega(n)}{\omega(n)} \right)^{\alpha}}{N} = 1
\]

A number is squarefree or quadratfrei if either it is the product of distinct primes or 1. That is, its prime factorization is of the form \( q_1 \cdots q_r \) donde the \( q_i \) \((i = 1, \ldots, r)\) \((r \geq 1)\) are the distinct primes. Let \( Q(x) \) be the number of quadratfrei numbers not exceeding \( x \). We have the following formula (see, for example, either [2] or [5]).

\[
Q(x) = \frac{6}{\pi^2} x + o(x)
\]

That is, the quadratfrei have positive density \( \frac{6}{\pi^2} \).

We need the following lemmas before our main theorem.

**Lemma 1.1** Let \( Q_{q_1 \cdots q_r}(x) \) the number of quadratfreis not exceeding \( x \) relatively primes to the quadratfrei \( q_1 \cdots q_r \). The following formula holds.

\[
Q_{q_1 \cdots q_r}(x) = \frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} x + o(x)
\]

Proof. See [4]. The lemma is proved.

A number is powerful or squareful if all the distinct primes in its prime factorization have multiplicity (or exponent) greater than 1. That is, the number \( q_1^{s_1} \cdots q_r^{s_r} \) is squareful if \( s_i \geq 2 \) \((i = 1, \ldots, r)\) \((r \geq 1)\).
Lemma 1.2 We have
\[ \sum_{q_1^{s_1} \cdots q_r^{s_r}} \frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} \frac{1}{q_1^{s_1} \cdots q_r^{s_r}} = 1 - \frac{6}{\pi^2} \]
where the sum run on all squareful numbers \( q_1^{s_1} \cdots q_r^{s_r} \).

Proof. We have
\[ \sum_{q_1^{s_1} \cdots q_r^{s_r}} \frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} \frac{1}{q_1^{s_1} \cdots q_r^{s_r}} = \prod_p \left( 1 + \frac{p}{1 + \frac{1}{p^2}} \right) = \prod_p \left( 1 + \frac{p}{p + 1 \frac{1}{p^2}} \right) - \frac{6}{\pi^2} = 1 - \frac{6}{\pi^2} \]
The lemma is proved.

Lemma 1.3 Let \( \alpha \) be an arbitrary but fixed positive real number. The following series converges.
\[ \sum_{q_1^{s_1} \cdots q_r^{s_r}} \frac{(\Omega(q_1^{s_1} \cdots q_r^{s_r}))^\alpha}{q_1^{s_1} \cdots q_r^{s_r}} \]
where the sum run on all squareful numbers \( q_1^{s_1} \cdots q_r^{s_r} \).

Proof. Let \( s_n \) be the \( n \)-th squareful number and let \( S(x) \) be the number of squareful numbers not exceeding \( x \). It is well-known the following asymptotic formula \( S(x) \sim c_1 \sqrt{x} \), where \( c_1 \) is a constant (see, for example, [3]). Therefore if \( x = s_n \) we obtain \( n = S(s_n) \sim c_1 \sqrt{s_n} \), that is, \( s_n \sim c_1 n^{2/3} \). On the other hand, we clearly have \( n \geq 2^{\Omega(n)} \), that is, \( \Omega(n) \leq \log \frac{n}{\log 2} = c_2 \log n \). Consequently \( \Omega(s_n) \leq c_3 \log s_n \sim c_3 \log n \). Now, the lemma follows by the Comparison Criterion since the series \( \sum \frac{\log n}{n^{2/3}} \) converges. The lemma is proved.

Theorem 1.4 Let \( \alpha \) be an arbitrary but fixed real number. The following limit holds.
\[ \lim_{N \to \infty} \frac{\sum_{n=1}^{N} \left( \frac{\Omega(n)}{\omega(n)} \right)^\alpha}{N} = 1 \]
(2)

Proof. Note that the intersection of two sets of density 1 is again a set of density 1. It is well-known (see either [1] or [2, Chapter XXII]) that the normal order of \( \Omega(n) \) is \( \log \log n \) and the normal order of \( \omega(n) \) is also \( \log \log n \).
Consequently given $\epsilon > 0$ there exists a set $C_\epsilon$ of density 1 depending of $\epsilon$ such that the following two inequalities hold.

\[(1 - \epsilon) \log \log n < \omega(n) < (1 + \epsilon) \log \log n \quad (n \in C_\epsilon) \quad (3)\]

\[(1 - \epsilon) \log \log n < \Omega(n) < (1 + \epsilon) \log \log n \quad (n \in C_\epsilon) \quad (4)\]

First, we shall prove the theorem when $\alpha < 0$. If $\alpha < 0$ then we can write $\alpha = -\beta$ where $\beta > 0$. Equations (3) and (4) give

\[1 - \epsilon' = \left( \frac{1 - \epsilon}{1 + \epsilon} \right)^\beta < \left( \frac{\omega(n)}{\Omega(n)} \right)^\beta \quad (5)\]

Let $A(N)$ be the number of elements in the set $C_\epsilon$ not exceeding $N$. Then we have

\[\lim_{N \to \infty} \frac{A(N)}{N} = 1 \quad (6)\]

since the set $C_\epsilon$ has density 1.

Equation (5) gives

\[\sum_{1 \leq n \leq N} \left( \frac{\omega(n)}{\Omega(n)} \right)^\beta \geq (1 - \epsilon')A(N) \quad (7)\]

Equations (6) and (7) give

\[\frac{\sum_{1 \leq n \leq N} \left( \frac{\omega(n)}{\Omega(n)} \right)^\beta}{N} \geq (1 - \epsilon') \frac{A(N)}{N} \geq (1 - \epsilon')(1 - \epsilon') \geq 1 - 2\epsilon' \quad (8)\]

On the other hand, since $\omega(n) \leq \Omega(n)$ for every $n$ we have

\[\left( \frac{\omega(n)}{\Omega(n)} \right)^\beta \leq 1\]

and consequently

\[\frac{\sum_{1 \leq n \leq N} \left( \frac{\omega(n)}{\Omega(n)} \right)^\beta}{N} \leq 1 \quad (9)\]

Equations (8) and (9) give limit (2), since $\epsilon' > 0$ can be arbitrarily small. Therefore the theorem is proved when $\alpha < 0$.

Now, suppose that $\alpha > 0$. Equations (3) and (4) give

\[\left( \frac{\Omega(n)}{\omega(n)} \right)^\alpha \leq \left( \frac{1 + \epsilon}{1 - \epsilon} \right)^\alpha = 1 + \epsilon' \quad (10)\]
Let \( q_1^{s_1} \cdots q_r^{s_r} \) be a fixed squareful number. Let us consider the numbers whose prime factorization is of the form \( q_1^{s_1} \cdots q_r^{s_r} q \) where \( q \) is a variable squarefree (including 1) relatively prime with \( q_1^{s_1} \cdots q_r^{s_r} \). The number of these numbers \( q_1^{s_1} \cdots q_r^{s_r} q \) not exceeding \( x \) will be denoted \( Q_{q_1^{s_1} \cdots q_r^{s_r}}(x) \) and by Lemma 1.1 we have

\[
Q_{q_1^{s_1} \cdots q_r^{s_r}}(x) = \frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} \frac{x}{q_1^{s_1} \cdots q_r^{s_r}} + o(x) = \rho_{q_1^{s_1} \cdots q_r^{s_r}} x + o(x) \tag{11}
\]

Since for every \( n \)

\[
\frac{\Omega(n)}{\omega(n)} \geq 1 \tag{12}
\]
we have (see (11))

\[
A_{q_1^{s_1} \cdots q_r^{s_r}}(x) = \sum_{q_1^{s_1} \cdots q_r^{s_r} q \leq x} \left( \frac{\Omega(q_1^{s_1} \cdots q_r^{s_r} q)}{\omega(q_1^{s_1} \cdots q_r^{s_r} q)} \right)^\alpha \geq \sum_{q_1^{s_1} \cdots q_r^{s_r} q \leq x} 1 = \rho_{q_1^{s_1} \cdots q_r^{s_r}} x + o(x)
\]

That is

\[
A_{q_1^{s_1} \cdots q_r^{s_r}}(x) \geq \rho_{q_1^{s_1} \cdots q_r^{s_r}} + o(1) \geq \rho_{q_1^{s_1} \cdots q_r^{s_r}} - 4\epsilon' \tag{13}
\]

Note that

\[
\left( \frac{\Omega(q_1^{s_1} \cdots q_r^{s_r} q)}{\omega(q_1^{s_1} \cdots q_r^{s_r} q)} \right)^\alpha = \left( \frac{\Omega(q_1^{s_1} \cdots q_r^{s_r} q) + \omega(q)}{\omega(q_1^{s_1} \cdots q_r^{s_r} q) + \omega(q)} \right)^\alpha \leq \left( \frac{\Omega(q_1^{s_1} \cdots q_r^{s_r})}{\omega(q_1^{s_1} \cdots q_r^{s_r})} \right)^\alpha \tag{14}
\]

since if \( A > B > 0 \) the function

\[
f(x) = \left( \frac{A + x}{B + x} \right)^\alpha \quad (x \geq 0)
\]
is strictly decreasing \((f'(x) < 0)\). Equations (10), (11) and (14) give

\[
A_{q_1^{s_1} \cdots q_r^{s_r}}(x) = \sum_{q_1^{s_1} \cdots q_r^{s_r} q \leq x} \left( \frac{\Omega(q_1^{s_1} \cdots q_r^{s_r} q)}{\omega(q_1^{s_1} \cdots q_r^{s_r} q)} \right)^\alpha \leq (1 + \epsilon') \left( \rho_{q_1^{s_1} \cdots q_r^{s_r}} x + o_1(x) \right)
\]

\[
+ \left( \frac{\Omega(q_1^{s_1} \cdots q_r^{s_r})}{\omega(q_1^{s_1} \cdots q_r^{s_r})} \right)^\alpha o_2(x) \tag{15}
\]

Note also that the intersection of a set of positive density \( \rho \) with a set of positive density 1 is again a set of positive density \( \rho \).

If \( N \) is sufficiently large we have

\[
o_1(1) \leq \epsilon', \quad o_2(1) \leq \left( \frac{\omega(q_1^{s_1} \cdots q_r^{s_r})}{\Omega(q_1^{s_1} \cdots q_r^{s_r})} \right)^\alpha \epsilon' \tag{16}
\]
Substituting (16) into (15) we obtain
\[
A_{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r}}(x) \leq \rho_{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r}} + \epsilon' \rho_{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r}} + \epsilon' + (\epsilon')^2 + \epsilon' \leq \rho_{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r}} + 4\epsilon'
\] (17)

Equations (13) and (17) give
\[
\lim_{x \to \infty} \frac{A_{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r}}(x)}{x} = \rho_{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r}}
\]

That is
\[
A_{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r}}(x) = \rho_{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r}} x + o(x)
\] (18)
since \(\epsilon' > 0\) can be arbitrarily small.

Equation (12) implies
\[
\sum_{n=1}^{N} \left( \frac{\Omega(n)}{\omega(n)} \right)^{\alpha} \geq 1
\] (19)

Let \(\epsilon > 0\). By Lemma 1.3 there exists a squareful \(A\) such that we have
\[
\sum_{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r} > A} \frac{\left( \Omega \left( q_{s_1}^{r_1} \cdots q_{s_r}^{r_r} \right) \right)^{\alpha}}{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r}} \leq \epsilon
\] (20)

On the other hand, Lemma 1.2 give us
\[
\sum_{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r} \leq A} \rho_{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r}} \leq 1 - \frac{6}{\pi^2}
\] (21)

Therefore, we have (see (1), (18) and (21))
\[
\sum_{n=1}^{N} \left( \frac{\Omega(n)}{\omega(n)} \right)^{\alpha} = \frac{6}{\pi^2} N + o(N) + \sum_{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r} \leq A} \left( \rho_{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r}} N + o(N) \right) + F(N)
\]
\[
= \frac{6}{\pi^2} N + \left( \sum_{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r} \leq A} \rho_{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r}} \right) N + o(N) + F(N)
\]
\[
\leq N + o(N) + F(N)
\] (22)

where (see (14) and (20))
\[
0 \leq F(N) = \sum_{A < q_{s_1}^{r_1} \cdots q_{s_r}^{r_r} \leq A_N} A_{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r}}(N)
\]
\[
\leq \sum_{A < q_{s_1}^{r_1} \cdots q_{s_r}^{r_r} \leq A_N} \left( \frac{\Omega \left( q_{s_1}^{r_1} \cdots q_{s_r}^{r_r} \right) \right)^{\alpha} \frac{N}{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r}} \right)
\]
\[
\leq N \sum_{A < q_{s_1}^{r_1} \cdots q_{s_r}^{r_r} \leq A_N} \left( \frac{\Omega \left( q_{s_1}^{r_1} \cdots q_{s_r}^{r_r} \right) \right)^{\alpha} \frac{N}{q_{s_1}^{r_1} \cdots q_{s_r}^{r_r}} \right)
\]
\[
\leq \epsilon N
\] (23)
In this equation $A_N$ is the greatest squareful not exceeding $N$.

Equations (22) and (23) give

$$\sum_{n=1}^{N} \frac{\Omega(n)}{\omega(n)}^\alpha \leq o(1) + \frac{F(N)}{N} + 1 \leq 1 + 2\epsilon \quad (N \geq N_\epsilon) \quad (24)$$

Finally, equations (19) and (24) give limit (2), since $\epsilon > 0$ can be arbitrarily small. The theorem is proved.

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References


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