The Locating-Chromatic Number of

Subdivision Firecracker Graphs

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Abstract

The locating-chromatic number of a graph was combined two graph concept, coloring vertices and partition dimension of a graph. In this paper, we discuss about locating-chromatic number of a subdivision firecracker graphs.

Keywords: graph, color code, locating-chromatic number

1. Introduction

The locating-chromatic number of a graph was introduced by Chartrand et al. [1] in 2002, with derived two graph concept, coloring vertices and partition dimension of a graph. Let $G = (V, E)$ be a connected graph and $c$ be a proper $k$-coloring of $G$ with color $1, 2, ..., k$. Let $\Pi = \{C_1, C_2, ..., C_k\}$ be a partition of $V(G)$ which is induced by coloring $c$. The color code $c_\Pi(v)$ of $v$ is the ordered $k$-tuple $(d(v, C_1), d(v, C_2), ..., d(v, C_k))$ where $d(v, C_i) = \min \{d(v, x) | x \in C_i\}$ for any $i$. If all distinct vertices of $G$ have distinct color codes, then $c$ is called $k$-locating coloring of $G$. The locating-chromatic number, denoted by $\chi_L(G)$, is the smallest $k$. 
such that $G$ has a locating $k$-coloring. A vertex $u \in S_i$, for some $i$, is dominant if $d(u, S_j) = 1$ for $j \neq i$.

Chartrand et al. [2] determined the locating chromatic number for some graph classes, such as on $P_n$ is a path of order $n \geq 3$ then $\chi_L(P_n) = 3$; for a cycle $C_n$ if $n$ is odd $\chi_L(C_n) = 3$, and if $n$ even $\chi_L(C_n) = 4$; for double star graph $(S_{a, b})$, $1 \leq a \leq b$ and $b \geq 2$, obtained $\chi_L(S_{a, b}) = b + 1$. Asmiati et al. [3] obtained the locating chromatic number of amalgamation of stars and furthermore, Asmiati [4] found for non homogeneous amalgamation of stars. Next, Asmiati [5] investigated the locating chromatic number for banana tree.

A Firecracker graph $F_{n,k}$, namely the graph obtained by the concatenation of $n$ stars $S_k$ by linking one leaf from each star. Let $V(F_{n,k}) = \{x_i, m_i, l_{ij} | i \in [1, n]; j \in [1, k - 2]\}$ and $E(F_{n,k}) = \{x_i, m_{i+1} | i \in [1, n]\} \cup \{x_i, m_i, l_{ij} | i \in [1, n]; j \in [1, k - 2]\}$. If we give subdivision one vertex $y_i$ in edge $x_i, m_i$, we denote $F_{n,k}^*$ with $n, k$ natural numbers.

Asmiati et al. [6] investigated the locating-chromatic number of firecracker graph $F_{n,k}$, for $n \geq 2$, $\chi_L(F_{n,k}) = 4$; for $k \geq 5, \chi_L(F_{n,k}) = k - 1$ for $2 \leq n \leq k - 1$ and $\chi_L(F_{n,k}) = k$ otherwise. Next, Asmiati [7] determined the locating chromatic number of $F_{n,k}^*$ with $n, k$ natural numbers. Besides that in this paper, we will discuss about the locating chromatic number of subdivision firecracker graphs.

The following theorem is basic to determine the lower bound of the locating chromatic of a graph. The set of neighbours of a vertex $s$ in $G$, denoted by $N(s)$.

**Theorem 1.1**: Chartrand et al. [1]

Let $G$ be a connected graph and $c$ is a locating coloring in $G$. If $u$ and $s$ are distinct vertices of $G$ such that $d(u,w)=d(s,w)$ for all $w \in V(G) - \{u, s\}$, then $c(u) \neq c(s)$. In particular, if $u$ and $s$ are adjacent to the same vertices in $G$ such that $N(u) \neq N(s)$, then $c(u) \neq c(s)$.

**Corollary 1.1**: Chartrand et al. [1]

If $G$ is a connected graph containing a vertex adjacent to $k$ leaves of $G$, then $\chi_L(G) \geq k + 1$. 
2. Main Results

In this section we will discuss the locating chromatic number of subdivision firecracker graphs, namely $F_{n,k}^{s*}$. $F_{n,k}^{s*}$ is a graph obtained from subdivision graph $F_{n,k}$ as much as $s \geq 2$ even vertices on each side $x_i, y_i$ and $y_im_i$ for every $i \in [1, n]$. Thus $x_iy_i$ and $y_im_i$ become a path for every $i \in [1, n]$. Let path $x_iy_i = \{x_i, a_{i1}, a_{i2}, ..., a_{ir}, y_i\}$ for every $r \in [1, s]$ and $s \geq 2$ even, path $y_im_i = \{y_i, b_{i1}, b_{i2}, ..., b_{ir}, m_i\}$ for every $r \in [1, s]$ and $s \geq 2$ even.

**Theorem 2.1.**

Let $F_{n,k}^{s*}$ be a subdivision firecracker graphs. Then,

i. $\chi_L(F_{n,k}^{s*}) = \left\{ \begin{array}{ll} k - 1, & 1 \leq n \leq k - 1 \\ k, & \text{otherwise.} \end{array} \right.$

ii. For $k \geq 5$

Proof:

(i) By Corollary 1.1, we have $\chi_L(F_{n,k}^{s*}) \geq 3$. For a contradiction, assume we have 3-locating coloring on $F_{n,k}^{s*}$ for $n \geq 2$, namely 1, 2, and 3. So, $\{c(m_1), c(l_{11}), c(l_{12})\} = \{c(m_2), c(l_{21}), c(l_{22})\} = \{1,2,3\}$. If we assign $c(m_1)$ is one of $\{1,2,3\}$ then color of $b_1$ is same with one of colors $l_{11}, l_{12}$. Therefore there are two vertices have the same color codes, a contrary. So, $\chi_L(F_{n,k}^{s*}) \geq 4$ for $n \geq 2$.

Next, we determine the upper bound of $F_{n,k}^{s*}$ for $n \geq 2$. Assign the 4-coloring $c$ on $F_{n,k}^{s*}$ as follows:

- $c(x_i) = 3$ for odd $i$ and $c(x_i) = 2$ for even $i$.
- $c(a_{ir}) = c(y_i)$ for odd $r$ and $c(a_{ir}) = c(x_i)$ for even $r$.
- $c(y_i) = 2$, for odd $i$ and $c(y_i) = 1$ for even $i$.
- $c(b_{ir}) = c(m_i)$ for odd $r$ and $c(a_{ir}) = c(y_i)$ for even $r$.
- $c(m_i) = 3$ for odd $i$ and $c(m_i) = 2$ for even $i$.
- For all vertices $l_{ij}$, define

$$c(l_{ij}) = \left\{ \begin{array}{ll} 4 & \text{if } i = 1, j = 1 \\ 1 & \text{if } i \geq 2, j = 1 \\ 2 & \text{for odd } i, j = 2 \\ 3 & \text{for even } i, j = 2 \end{array} \right.$$

The coloring $c$ will create the partition $\Pi$ on $V(F_{n,k}^{s*})$. We shall show that the color codes of all vertices in $F_{n,k}^{s*}$ are different. For odd $i$, we have $c_\Pi(x_i) = (2,1,0,i + 2 + 2s)$ and for even $i$, $c_\Pi(x_i) = (1,0,1,i + 2 + 2s)$. For $y_1$, $c_\Pi(y_1) = (3 + s,0,1,2 + s)$ and for $i \geq 2$, we have $c_\Pi(y_i) = (3,0,1,i + 3 + 3s)$ for odd $i$, $c_\Pi(y_i) = (0,1,2,i + 3 + 3s)$ for even $i$. For $m_i$, we have $c_\Pi(m_i) = (4,1,0,1)$ and for $i \geq 2$, we have $c_\Pi(m_i) = (1,1,0,i + 4 + 4s)$ for odd $i$, $c_\Pi(m_i)$ =
(1,0,1,i + 4 + 4s) for even i. For vertices \( v_{ij} \), we have \( c_{\Pi}(l_{11}) = (5 + 3s, 2,1,0) \) and \( c_{\Pi}(l_{12}) = (5 + 3s, 0,1,2) \). For \( i \geq 2 \), \( c_{\Pi}(l_{11}) = (0,1,2,i + 5 + 4s) \), \( c_{\Pi}(l_{ij}) = (0,2,1,i + 5 + 4s) \) for odd \( i \), \( c_{\Pi}(l_{ij}) = (2,1,0,i + 5 + 4s) \) for even \( i \). For \( c(a_{ir}) = c(y_i) \) for odd \( r \) and \( c(a_{ir}) = c(x_i) \) for even \( r \), for \( c(b_{ir}) = c(m_i) \) for odd \( r \) and \( c(b_{ir}) = c(y_i) \) for even \( r \), for some \( r \in [1,s] \) and \( s \geq 2 \) even.

Let \( u, v \in V(F_{n,k}^{s+}) \) and \( c(u) = c(v) \) then:

- If \( u = a_{ih} \) and \( v = a_{jl} \) for some \( i,j,h,l \) and \( i \neq j \), then \( c_{\Pi}(u) \neq c_{\Pi}(v) \).
- If \( u = b_{ih} \) and \( v = b_{jl} \) for some \( i,j,h,l \) and \( i \neq j \), then \( c_{\Pi}(u) \neq c_{\Pi}(v) \).
- If \( u = a_{ih} \) and \( v = b_{jl} \) for some \( i,j,h,l \) and \( i \neq j \), then \( c_{\Pi}(u) \neq c_{\Pi}(v) \).

Since the color codes of all vertices in \( F_{n,k}^{s+} \) are different, thus \( c \) is locating chromatic coloring. So \( \chi_{L}(F_{n,k}^{s+}) \leq 4, n \geq 2 \). As a result, we have \( \chi_{L}(F_{n,k}^{s+}) = 4 ; n \geq 2 \).

(ii) Next, we will show that for \( k \geq 5 \), \( \chi_{L}(F_{n,k}^{s+}) = k \) if \( n \geq k \) and \( \chi_{L}(F_{n,k}^{s+}) = k - 1 \), if \( 1 \leq n \leq k - 1 \). To show this, let we consider the following two cases:

**Case 1.** For \( k \geq 5 \) and \( 1 \leq n \leq k - 1 \).

Since each vertex \( m_i \) is adjacent to \((k - 2)\) leaves, by Corollary 1.1, we have \( \chi_{L}(F_{n,k}^{s+}) \geq k - 1 \). Next, we will show that \( \chi_{L}(F_{n,k}^{s+}) \leq k - 1 \) for \( k \geq 5 \) and \( n \leq k - 1 \). Define a \((k - 1)\)-coloring \( c \) of \( F_{n,k}^{s+} \) as follows. Assign \( c(m_i) = i \), for \( i \in [1,n] \) and all the leaves: \( \{l_{ij} \mid j = 1,2,...,k-2\} \) by \( \{1,2,...,k-1\}\{i\} \) for any \( i \). However \( c(y_i) = 2 \), for odd \( i \) and 1 for even \( i \), \( c(x_i) \neq c(m_i) \) for \( i \in [1,2,...,k-1] \). For \( c(a_{ir}) = c(y_i) \) for odd \( r \) and \( c(a_{ir}) = c(x_i) \) for even \( r \), for every \( r \in [1,s] \) and \( s \geq 2 \) even. As a result, coloring \( c \) will create a partition \( \Pi = \{U_1,U_2,...,U_{k-1}\} \) on \( V(F_{n,k}^{s+}) \), where \( U_i \) is the set of all vertices with color \( i \).

We show that the color codes for all vertices in \( F_{n,k}^{s+} \) for \( k \geq 5 \) and \( n \leq k - 1 \) are different. Let \( u, v \in V(F_{n,k}^{s+}) \) and \( c(u) = c(v) \). Then, consider the following cases:

- If \( u = l_{ij} \), \( v = l_{jl} \) for some \( i, j, h, l \) and \( i \neq j \), then \( c_{\Pi}(u) \neq c_{\Pi}(v) \) since \( d(u,U_l) \neq d(v,U_l) \).
- If \( u = l_{ih} \), \( v = m_j \) for some \( i, j, h \) and \( i \neq j \), then \( u \) must be dominant vertex and \( v \) is not. So \( c_{\Pi}(u) \neq c_{\Pi}(v) \).
- If \( u = l_{ih} \), \( v = y_j \) for some \( i, j, h \) and \( i \neq j \), then there exactly one set in \( \Pi \) which has the distance 1 from \( u \) and there is at least two set in \( \Pi \) which has the distance 1 from \( v \). Thus \( c_{\Pi}(u) \neq c_{\Pi}(v) \).
- If \( u = l_{ih} \), \( v = x_j \) for some \( i, j, h \) and \( i \neq j \), then there exactly one set in \( \Pi \) which has the distance 1 from \( u \) and there is at least two set in \( \Pi \) which has the distance 1 from \( v \). Thus \( c_{\Pi}(u) \neq c_{\Pi}(v) \).
- If \( u = l_{ij} \), \( v = a_{jh} \) for some \( i, j, h \) and \( i \neq j \), then there exactly one set in \( \Pi \) which has the distance 1 from \( u \) and there is at least two set in \( \Pi \) which has the distance 1 from \( v \). Thus \( c_{\Pi}(u) \neq c_{\Pi}(v) \).
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- If \( u = m_i \), \( v = y_j \) for some \( i, j \) and \( i \neq j \), then \( u \) must be dominant vertex and \( v \) is not. So \( c_\Pi(u) \neq c_\Pi(v) \).
- If \( u = m_i \), \( v = x_j \) for some \( i, j \) and \( i \neq j \), then \( u \) must be dominant vertex and \( v \) is not. So \( c_\Pi(u) \neq c_\Pi(v) \).
- If \( u = m_i \), \( v = a_{ijh} \) for some \( i, j, h \) and \( i \neq j \), then \( u \) must be dominant vertex and \( v \) is not. So \( c_\Pi(u) \neq c_\Pi(v) \).
- If \( u = x_i \) and \( v = x_j \) for some \( i, j \) and \( i \neq j \), then \( c_\Pi(u) \neq c_\Pi(v) \) since \( d(u, U_i) \neq d(v, U_j) \).
- If \( u = y_i \) and \( v = a_{jh} \) for some \( i, j, h \) and \( i \neq j \), then \( c_\Pi(u) \neq c_\Pi(v) \) since \( d(u, U_i) \neq d(v, U_j) \).
- If \( u = x_i \) and \( v = x_j \) then \( i = 1 \) and \( j = n \). So \( c_\Pi(u) \neq c_\Pi(v) \).
- If \( u = a_{ih} \) and \( v = a_{il} \) for some \( i, j, h, l \) and \( i \neq j \), then \( c_\Pi(u) \neq c_\Pi(v) \).

For all the above cases, we can see that the color codes of all vertices in \( F_{n,k}^{SS} \) for \( k \geq 5, n \leq k - 1 \) are different. Thus \( \chi_d(F_{n,k}^{SS}) \leq k - 1 \) for \( 1 \leq n \leq k - 1 \). So, we have, \( \chi_d(F_{n,k}^{SS}) = k - 1 \) for \( 1 \leq n \leq k - 1 \).

**Case 2.** For \( k \geq 5 \) and \( n \geq k \).

Since each vertex \( m_i \) is adjacent to \( (k - 2) \) leaves, by Corollary 1.1, we have \( \chi_d(F_{n,k}^{SS}) \geq k - 1 \) for \( n \geq k \). For a contradiction, assume we have \( (k - 1) \)-locating coloring \( c \) on \( F_{n,k}^{SS} \) for \( k \geq 5 \) and \( n \geq k \). Since \( n \geq k \), then there are \( i, j, i \neq j \) such that \( \{c(l_{it}) \mid t \in [1, k - 2]\} = \{c(l_{jp}) \mid p \in [1, k - 2]\} \). Therefore the color codes of \( m_i \) and \( m_j \) are the same, a contradiction. As a result, \( \chi_d(F_{n,k}^{SS}) \geq k \) for \( k \geq 5 \) and \( n \geq k \).

Next, we determine the upper bound of \( F_{n,k}^{SS} \) for \( k \geq 5, \ n \geq k \). To show that \( \chi_d(F_{n,k}^{SS}) \leq k, \ k \geq 5 \) and \( n \geq k \), consider the locating coloring \( c \) on \( F_{n,k}^{SS} \) as follows:

- \( c(x_i) = 1 \) for odd \( i \) and \( c(x_i) = 3 \) for even \( i \).
- \( c(a_{ir}) = c(y_i) \) for odd \( r \) and \( c(a_{ir}) = c(x_i) \) for even \( r \).
- \( c(y_i) = 2 \), for every \( i \).
- \( c(b_{ir}) = c(m_i) \) for odd \( r \) and \( c(b_{ir}) = c(y_i) \) for even \( r \).
- \( c(m_i) = 1 \), for every \( i \).
- If \( A = \{1, 2, ..., k\} \), define
  \[
  \{c(l_{ij}) \mid j = 1, 2, ..., k - 2\} = \begin{cases} A \setminus \{1, k - 1\} & \text{if } i = 1 \\ A \setminus \{1, k\} & \text{otherwise}. \end{cases}
  \]

The coloring \( c \) will create a partition \( V(F_{n,k}^{SS}) \). We shall show that the color codes of all vertices in \( F_{n,k}^{SS} \) are different.

\[
c_\Pi(x_1) = \begin{cases} 0 & \text{for } i^{th} \text{ component} \\ 1 & \text{for } 2^{nd} \text{ and } 3^{rd} \text{ component} \\ 4 + 4s & \text{for } (k - 1)^{th} \text{ component} \\ 3 + 2s & \text{otherwise}. \end{cases}
\]

For \( i \geq 2 \) odd
\[ c_\Pi(x_i) = \begin{cases} 0 & \text{for } i^{th} \text{ component} \\ 1 & \text{for } 2^{nd} \text{ and } 3^{rd} \text{ component} \\ i + 2 + 2s & \text{for } k^{th} \text{ component} \\ 3 + 2s & \text{otherwise} \end{cases} \]

For \( i \geq 2 \) even

\[ c_\Pi(y_1) = \begin{cases} 1 & \text{for } i^{th} \text{ component} \\ 0 & \text{for } 3^{rd} \text{ component} \\ i + 2 + 2s & \text{for } k^{th} \text{ component} \\ 3 + 2s & \text{otherwise} \end{cases} \]

For \( i \geq 2 \) odd

\[ c_\Pi(y_i) = \begin{cases} 1 & \text{for } 1^{st} \text{ component} \\ 0 & \text{for } 2^{nd} \text{ component} \\ i + 3 + s & \text{for } k^{th} \text{ component} \\ 2 + s & \text{otherwise} \end{cases} \]

For \( i \geq 2 \) even

\[ c_\Pi(m_1) = \begin{cases} 1 & \text{for } 1^{st} \text{ and } 3^{rd} \text{ component} \\ 0 & \text{for } 2^{nd} \text{ component} \\ i + 3 + s & \text{for } k^{th} \text{ component} \\ 2 + s & \text{otherwise} \end{cases} \]

For \( i \geq 2 \) even

\[ c_\Pi(m_i) = \begin{cases} 0 & \text{for } 1^{st} \text{ component} \\ i + 4 + 4s & \text{for } k^{th} \text{ component} \\ 1 & \text{otherwise} \end{cases} \]

For \( j = 1, 2, \ldots, k - 2 \)

\[ c_\Pi(t_{ij}) = \begin{cases} 1 & \text{for } 1^{st} \text{ component} \\ 0 & \text{for } j^{th} \text{ component} \\ 7 + 4s & \text{for } (k - 1)^{th} \text{ component} \\ 2 & \text{otherwise} \end{cases} \]
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\[
c_{\Pi}(t_{1k-2}) = \begin{cases} 
1 & \text{for } 1^{st} \text{ component} \\
7 + 4s & \text{for } (k-1)^{th} \text{ component} \\
0 & \text{for } k^{th} \text{ component} \\
2 & \text{otherwise}.
\end{cases}
\]

For \( i \neq 1, j = 1, 2, ..., k-2 \)

\[
c_{\Pi}(t_{ij}) = \begin{cases} 
1 & \text{for } 1^{st} \text{ component} \\
0 & \text{for } j^{th} \text{ component} \\
i + 5 + 4s & \text{for } k^{th} \text{ component} \\
2 & \text{otherwise}.
\end{cases}
\]

Let \( c(a_{ir}) = c(y_i) \) and \( c(b_{ir}) = c(m_i) \) for odd \( r \); \( c(a_{ir}) = c(x_i) \) and \( c(b_{ir}) = c(y_i) \) for even \( r \), \( r \in [1, s] \) and \( s \geq 2 \) even. Let \( u, v \in V(F_{n,k}^{s*}) \) and \( c(u) = c(v) \) then:

- If \( u = a_{ih} \) and \( v = a_{jl} \) for some \( i, j, h, l \) and \( i \neq j \), then \( c_{\Pi}(u) \neq c_{\Pi}(v) \).
- If \( u = b_{ih} \) and \( v = b_{jl} \) for some \( i, j, h, l \) and \( i \neq j \), then \( c_{\Pi}(u) \neq c_{\Pi}(v) \).
- If \( u = a_{ih} \) and \( v = b_{jl} \) for some \( i, j, h, l \) and \( i \neq j \), then \( c_{\Pi}(u) \neq c_{\Pi}(v) \).

Since the color codes of all vertices are different, thus \( c \) is locating chromatic coloring in \( F_{n,k}^{s*} \), so \( \chi_{L}(F_{n,k}^{s*}) \leq k, n \geq k \). Thus, we have \( \chi_{L}(F_{n,k}^{s*}) = k \) for \( n \geq k \).

![Fig.(1): A minimum locating coloring of \( F_{6,5}^{2*} \)](image)

3. Conclusion

The locating chromatic number of subdivision firecracker graphs, namely \( F_{n,k}^{s*} \) is 4 for \( n \geq 2 \), whereas for \( k \geq 5 \) and \( 1 \leq n \leq k - 1 \), we have \( \chi_{L}(F_{n,k}^{s*}) = k - 1 \) and \( k \) for otherwise.
References


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