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Some Properties of Hamiltonian Semirings

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Abstract

In this paper, properties for Hamiltonian semirings are studied. We concluded that subsemirings and all homomorphic images of a Hamiltonian semiring are Hamiltonian semirings. Hamiltonian semirings satisfying the congruence extension property are characterized by the strong congruence extension property. Finally, we concluded that if $R \times R$ is a Hamiltonian semiring, then R has strong congruence extension property

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1 Introduction

Many experts and scholars have studied the congruence extension property and the strong congruence extension property for algebras. Relations among the various extensions are explored. Related results see [2, 3, 5, 6]. An algebra A is Hamiltonian if every its subalgebra is a class of some congruence on A. A variety $\mathscr{V}$ is Hamiltonian if each $A \in \mathscr{V}$ has this property.

A semigroup has the congruence extension property (CEP) provided that each congruence on each subsemigroup can be extended to the semigroup. This property, along with the ideal extension property and the group congruence extension property are studied in [4]. Note that others semigroup-theoretical methods were used in the study of semirings. In this paper, we obtained some properties about Hamiltonian semirings. Whether each of this properties is productive, hereditary or preserved by homomorphisms was determined.

However, if R is a Hamiltonian semiring, $R \times R$ may not be. That is, the class of all Hamiltonian semirings is not a variety. And we defined the strong congruence extension property (SCEP) for semirings. Connections between Hamiltonian semirings, CEP and SCEP were characterized. We concluded that if $R \times R$ is a Hamiltonian semiring, then R has SCEP.

The set of all congruences on R is notated by $Con(R)$. We denote the equality congruence on R by Δ_R. Let ρ be a relation on a semiring R. We can now easily obtain the following characterization of $\rho^\sharp$, the congruence on R generated by ρ.

Proposition 1.1. *Let ρ be a relation on a semiring R. Then $(a, b) \in \rho^\sharp$ if and only if either $a = b$ or for some n in N there is a sequence*

$$a = u_0 + x_0a_0y_0 + v_0 \to u_0 + x_0b_0y_0 + v_0 = u_1 + x_1a_1y_1 + v_1 \to$$
$$u_1 + x_1b_1y_1 + v_1 = u_2 + x_2a_2y_2 + v_2 \to \cdots \to u_n + x_nb_ny_n + v_n = b,$$

in which, for each $i \in \{0, 1, 2, ..., n\}$, $(a_i, b_i) \in \rho \cup \rho^{-1}$ and $u_i, v_i \in R^0$, $x_i, y_i \in R^1$.

2 Congruence Extension Property

Definition 2.1. We say that a semiring R has the congruence extension property (CEP) if every congruence ρ on any subsemiring S of R is the restriction of some congruence θ on R, i.e. $\theta \cap (S \times S) = \rho$.

Definition 2.2. Let R be a semiring. We say R is a Hamiltonian semiring if every its subsemiring is a class of some congruence on R.

Example 2.3. Let $R = \{1, 2, 3\}$ with Cayley table:

+	1	2	3
1	1	3	3
2	3	2	3
3	3	3	3

·	1	2	3
1	3	2	3
2	2	2	2
3	3	2	3

It can be shown by patient calculation that every its subsemiring is a class of some congruence on R. That is, R is a Hamiltonian semiring.

Proposition 2.4. *Every subsemiring of a Hamiltonian semiring is Hamiltonian.*

Proof. Suppose that R is a Hamiltonian semiring. If H is a subsemiring of R and if K is a subsemiring of H, then there exists a congruence θ of R such that $K = \theta_k (k \in K)$. Let $\rho = \theta \cap (H \times H)$. We can show easily that ρ is a congruence on H.

Now, we show that $K = \rho_k$. Let $x \in K$. Then $(x, k) \in \theta$ and $x, k \in H$. Therefore, $(x, k) \in \rho$. Then $K \subseteq \rho_k$. On the other hand, let $x \in \rho_k$. Then $(x, k) \in \theta$ and $x \in \theta_k$. Therefore, $\rho_k \subseteq K$. □

Proposition 2.5. *The homomorphic image of a Hamiltonian semiring is Hamiltonian.*

Proof. Suppose that R is a Hamiltonian semiring. Let $\phi : R \to S$ be a homomorphism of R onto a semiring S. Suppose K is a subsemiring of S and define $M = \{x \in R : \phi(x) \in K\}$, then M is a subsemiring of R. By assumption, R is a Hamiltonian semiring. There exists a congruence ρ on R such that $M = \rho_m(m \in M)$. Define $\theta = \{(\phi(x), \phi(y)) \in S \times S : (x, y) \in \rho\}$. Then θ is a congruence on S.

Now, we show that $K = \theta_k(k \in K)$. Let $a \in K$. Then there exist $x, y \in M$ such that $\phi(x) = a$, $\phi(y) = k$. Therefore, $(x, y) \in \rho$ which implies $(a, b) \in \theta$. So $K \subseteq \theta_k$. On the other hand, let $a \in \theta_k$. Then $(a, k) \in \theta$. Therefore, there exist $x \in R$, $y \in M$ such that $\phi(x) = a$, $\phi(y) = k$. Then $(x, y) \in \rho$. So $x \in M$ and $a \in K$. It follows that S is Hamiltonian. □

Corollary 2.6. *The quotient semiring of a Hamiltonian semiring is Hamiltonian.*

Example 2.7. Let R be the semiring of example 2.3, i.e. $R = \{1, 2, 3\}$ with Cayley table:

+	1	2	3
1	1	3	3
2	3	2	3
3	3	3	3

·	1	2	3
1	3	2	3
2	2	2	2
3	3	2	3

Rename the elements of $R \times R$ according to the the following scheme:

$$\begin{array}{lll} 1 = (1, 1) & 2 = (2, 1) & 3 = (3, 1) \\ 4 = (1, 2) & 5 = (2, 2) & 6 = (3, 2) \\ 7 = (1, 3) & 8 = (2, 3) & 9 = (3, 3). \end{array}$$

Consider the subsemiring $T = \{1, 9\}$ and $\rho = \{(1, 9), (9, 1)\} \cup \triangle_{R\times R}$. But $(1, 9) + (7, 7) = (7, 9)$, so ρ is not a congruence on $R \times R$, which is a contradiction. Then T is not a class of any congruence on $R \times R$.

Definition 2.8. We say that a semiring R has the strong congruence extension property (SCEP) if for every subsemiring S of R and each $\theta \in Con(S)$, there exists $\phi \in Con(R)$ with $\theta_s = \phi_s$ for each $s \in S$.

Theorem 2.9. *Let R be a semiring. Then R satisfies SCEP if and only if R is Hamiltonian and R satisfies CEP.*

Proof. Suppose that R satisfies SCEP and let S be a subsemiring of R. Put $\rho = S \times S$. By SCEP, there exists a congruence θ of R such that $S = \rho_s = \theta_s$ for each $s \in S$. Then R is Hamiltonian, it also satisfies CEP.

Now suppose that R is a Hamiltonian semiring and satisfies CEP. Let S be a subsemiring of R and ρ be a congruence on S. By CEP, there exists a congruence θ on R such that $\theta \cap (S \times S) = \rho$. Since R is a Hamiltonian semiring, S is a class of some congruence ϕ on R. Then $(\theta \cap \phi)_s = \theta_s \cap \phi_s = \theta_s \cap S = \rho_s$ for each $s \in S$, thus R satisfies SCEP. □

As we know, every Hamiltonian variety has CEP. However, the class of all Hamiltonian semirings is not a variety by example 2.7. That is to say, a Hamiltonian semiring may not have CEP.

Example 2.10. Let $R = \{1, 2, 3, 4\}$ with Cayley table:

+	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1
4	1	1	1	3

·	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	3
4	1	1	3	4

By calculating we can see that $(R, +, \cdot)$ is a Hamiltonian semiring. To see that R does not have CEP, consider the subsemiring $S = \{1, 2, 3\}$ and a congruence $\rho = \{(2, 3), (3, 2)\} \cup \triangle_S$ on S, suppose σ is an extension of ρ on R. Since $(2, 3) \in \rho$, also $(2, 3) \in \sigma$. Further $(4, 4) \in \sigma$, thus also $(1, 3) = (2, 3)(4, 4) \in \sigma$. Since $1, 3 \in S$, we obtain $(1, 3) \in \sigma|_S$, but $(1, 3) \notin \rho$, which is a contradiction.

Theorem 2.11. *If R is a semiring such that $R \times R$ is a Hamiltonian semiring, then R has SCEP.*

Proof. Since R is isomorphic to the subsemiring $M = \{(a, a) : a \in R\}$ of $R \times R$, then R is a Hamiltonian semiring. Let S be a subsemiring of R and θ a congruence on S. Let us denote $\overline{\theta}$ the congruence on R generated by θ. Obviously, $\theta \subseteq \overline{\theta} \cap (S \times S)$.

Now, we show that $\overline{\theta} \cap (S \times S) \subseteq \theta$. Let $(a, b) \in \overline{\theta} \cap (S \times S)$. Then we have either $a = b$ or for some n in N there is a sequence

$$a = u_0 + x_0 a_0 y_0 + v_0 \to u_0 + x_0 b_0 y_0 + v_0 = u_1 + x_1 a_1 y_1 + v_1 \to$$
$$u_1 + x_1 b_1 y_1 + v_1 = u_2 + x_2 a_2 y_2 + v_2 \to \cdots \to u_n + x_n b_n y_n + v_n = b,$$

in which, for each $i \in \{0, 1, 2, ..., n\}$, $(a_i, b_i) \in \theta$ and $u_i, v_i \in R^0$, $x_i, y_i \in R^1$. Since θ is a subsemiring of $S \times S$, then θ is a subsemiring of $R \times R$. By $R \times R$ is Hamiltonian, there exists a congruence ϕ on $R \times R$ such that $\theta = \phi_{(a_i, b_i)}$. Since $(a_i, a_i) \in \theta$ and $(a_i, b_i) \in \theta$, then $(a_i, a_i)\phi(a_i, b_i)$. Therefore $(u_i + x_i a_i y_i + v_i, u_i +$

$x_ia_iy_i+v_i)\phi(u_i+x_ia_iy_i+v_i, u_i+x_ib_iy_i+v_i)(i\in\{0,1,2,...,n\})$. In particular, $(a,a)=(u_0+x_0a_0y_0+v_0, u_0+x_0a_0y_0+v_0)\phi(u_0+x_0a_0y_0+v_0, u_0+x_0b_0y_0+v_0)$, which follows that $(u_0+x_0a_0y_0+v_0, u_0+x_0b_0y_0+v_0)\in\phi_{(a,a)}=\theta$. Iteratively, we deduce $(u_i+x_ia_iy_i+v_i, u_i+x_ia_iy_i+v_i)\phi(u_i+x_ia_iy_i+v_i, u_i+x_ib_iy_i+v_i)\in\theta(i\in\{0,1,2,...,n\})$. That is, $(a,b)\in\theta$. It follows that R has CEP. From theorem 2.9, we conclude that R has SCEP. □

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