A Short Note on Bounds of Euler’s Number $e$

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Abstract

Using binomial theorem and basic pre-calculus algebra knowledge, we present a simple proof of the well-known bounds of Euler’s number $e$.

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1. Introduction

Euler’s number $e$, which is an irrational number whose value is $2.7182818...$, plays an important role in both theoretical and applied mathematics. Many Calculus textbooks, for example, Tan’s Applied Calculus [2], introduce Euler’s number $e$ in the chapter of Exponential and Logarithmic Functions.

It is well-known that Euler’s number $e$ can be defined as (see, for example, [1, 3, 4, 5] and their references):

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$  \hspace{1cm} (1.1)

It can also be shown that $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n+1}$ (see, for example, [5]).

Readers can easily construct or exam the following table and it may help to convince the plausibility of these two definitions of $e$. 
\[ n \quad \begin{array}{|c|c|c|} \hline \quad (1 + \frac{1}{n})^n \quad \quad (1 + \frac{1}{n})^{n+1} \quad \hline 1 \quad 2.000000 \quad 4.000000 \quad \hline 2 \quad 2.250000 \quad 3.375000 \quad \hline 5 \quad 2.488320 \quad 2.985984 \quad \hline 10 \quad 2.593742 \quad 2.853117 \quad \hline 100 \quad 2.704814 \quad 2.731862 \quad \hline 1000 \quad 2.716924 \quad 2.719641 \quad \hline 10,000 \quad 2.718146 \quad 2.718418 \quad \hline 100,000 \quad 2.718268 \quad 2.718295 \quad \hline 100,000,0 \quad 2.718282 \quad 2.718282 \quad \hline \end{array} \]

2. Main Results

In this section, we would like to provide a simple proof of the following two well-known inequalities, using binomial theorem and basic pre-calculus algebra knowledge.

For all integers \( n \geq 1 \),
\[
\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1} < e. \tag{1.2}
\]

For all integers \( m \geq 2 \),
\[
e < \left(1 + \frac{1}{m}\right)^{m+1} < \left(1 + \frac{1}{m-1}\right)^m. \tag{1.3}
\]

Proof. Using the binomial theorem, we can have
\[ \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k \]
\[ = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left(\frac{1}{n}\right)^k \]
\[ = \sum_{k=0}^{n} \frac{n^k(n-k)!}{k!} \cdot \frac{1}{k!} \]
\[ = \sum_{k=0}^{n} \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1) \cdot 1}{n^k k!} \]
\[ = \sum_{k=0}^{n} \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-(k-1)}{n} \cdot \frac{1}{k!} \]
\[ = \sum_{k=0}^{n} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \cdot \frac{1}{k!} \]
\[ \leq \sum_{k=0}^{n} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n+1}\right) \cdot \frac{1}{k!} \]
\[ < \sum_{k=0}^{n+1} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n+1}\right) \cdot \frac{1}{k!} \]

Since \( \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) \cdot \frac{1}{k!} \) as shown in above steps, replace \( n \) by \( n + 1 \) to get
\[ \left(1 + \frac{1}{n+1}\right)^{n+1} = \sum_{k=0}^{n+1} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n+1}\right) \cdot \frac{1}{k!} \].

Therefore, \( \left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1} \). i.e., \( \left\{ \left(1 + \frac{1}{n}\right)^n \right\}, \; n = 1, 2, 3, \ldots \) is an increasing sequence.

Since \( e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \), \( \left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1} < e \), which proves (1.2).

On the other hand, for integers \( m \geq 2 \) and any \( x > 0 \), applying the binomial theorem again, we can have \( (1+x)^m > 1 + mx \).

Therefore, for integers \( m \geq 2 \),
\[
\left(1 + \frac{1}{m-1}\right)^m = \left(\frac{m}{m-1}\right)^m = \left(\frac{m^2}{m^2-1}\right)^m \left(\frac{m+1}{m}\right)^m
\]
\[
= \left(1 + \frac{1}{m^2-1}\right)^m \left(1 + \frac{1}{m}\right)^m
> \left(1 + m \cdot \frac{1}{m^2-1}\right)^m \left(1 + \frac{1}{m}\right)^m
> \left(1 + m \cdot \frac{1}{m^2}\right)^m \left(1 + \frac{1}{m}\right)^m
= \left(1 + \frac{1}{m}\right)^{m+1}
\]
i.e. \(\left(1 + \frac{1}{m}\right)^{m+1}, \ m = 1, 2, 3, \cdots\) is a decreasing sequence.

Since \(e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{m+1}, \ e < \left(1 + \frac{1}{m}\right)^{m+1} < \left(1 + \frac{1}{m-1}\right)^m\), which proves (1.3).

References


[2] T. N. T. Goodman, Maximum products and \(\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e\), Amer. Math. Monthly, 93 (1986), 638–640. https://doi.org/10.2307/2322326


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