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Some Reviews on Ranks of Upper Triangular Block Matrices over a Skew Field

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Abstract

The aim of this article is to go over necessary and sufficient conditions that $r\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) = r(A)+r(B)$, which is a rank equation of 2×2 upper triangular block matrices over a skew field.

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1 Introduction

The basic method to find a rank of a matrix over a field is to apply row operations on a matrix to get the associated row ehcelon matrix which is an upper triangular matrix. This method can also be done in the case of a matrix over a skew field. From this reason, we are interested in finding a rank of an upper triangular block matrix. In particular, we focus on the case that the rank of an upper triangular block matrix is eqaul to a sum of ranks of matrices on the main diagonal of the block matrix. Matsaglia and Styan in [3] provided many important inequalities on rank of matrices over a field. In this work, we give some results similar to those in their work but over a skew field K instead. We shall start with basic definitions and theorems for matrices over a skew field (for more details see [2]). Let K denote a skew field, $K^{m\times n}$ denote the set of all $m\times n$ matrices with all entries in K, $\overline{0}_{m\times n}$ denote the $m\times n$ zero matrix, and I_n denote the $n\times n$ identity matrix for any $m,n\in\mathbb{N}$.

Definition 1.1. *Let $A\in K^{m\times n}$. The subspace of $K^{1\times n}$ spanned by the row vectors of A is called the row space of A, denoted by $\mathcal{R}(A)$, and the subspace of $K^{m\times 1}$ spanned by the column vectors of A is called the column space of A, denoted by $\mathcal{C}(A)$. The dimension of the row space of a matrix A is called the row rank of A, and the dimension of the column space of a matrix A is called the column rank of A.*

Theorem 1.2. *Let A be an arbitrary $m\times n$ matrix over a skew field K. Then the row rank of A and the column rank of A are equal, denoted by $r(A)$.*

Proof. Let $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}$.

Denote $R_i=(a_{i1},a_{i2},\dots,a_{in})$ the i^{th} row of A for all $i=1,2,\dots,m$.
Suppose the row rank of A equals r. Then there exist r rows of A forming a basis of the row space of A. Without loss of generality, say $R_{i_1},R_{i_2},\dots,R_{i_r}$ where $R_{i_k}=(a_{i_k1},a_{i_k2},\dots,a_{i_kn})$ for all $k=1,2,\dots,r$ and $i_1<i_2<\ldots<i_r$. Then each R_i of A is a linear combination of $R_{i_1},R_{i_2},\dots,R_{i_r}$.
We obtain

$$\begin{aligned} R_1 &= \alpha_{11}R_{i_1}+\alpha_{12}R_{i_2}+\cdots+\alpha_{1r}R_{i_r}\\ R_2 &= \alpha_{21}R_{i_1}+\alpha_{22}R_{i_2}+\cdots+\alpha_{2r}R_{i_r}\\ &\vdots\\ R_m &= \alpha_{m1}R_{i_1}+\alpha_{m2}R_{i_2}+\cdots+\alpha_{mr}R_{i_r}.\end{aligned}$$

Substitute each R_i by $(a_{i1},a_{i2},\dots,a_{in})$ and each R_{i_k} by $(a_{i_k1},a_{i_k2},\dots,a_{i_kn})$. We obtain

$$a_{ij}=\alpha_{i1}a_{i_1j}+\alpha_{i2}a_{i_2j}+\cdots+\alpha_{ir}a_{i_rj}$$

for all $i=1,2,\dots,m$ and $j=1,2,\dots,n$. We rewrite that

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} = \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \vdots \\ \alpha_{m1} \end{bmatrix} a_{i_1 j} + \begin{bmatrix} \alpha_{12} \\ \alpha_{22} \\ \vdots \\ \alpha_{m2} \end{bmatrix} a_{i_2 j} + \cdots + \begin{bmatrix} \alpha_{1r} \\ \alpha_{2r} \\ \vdots \\ \alpha_{mr} \end{bmatrix} a_{i_r j}.$$

Note that for each k, $\begin{bmatrix} \alpha_{1k} \\ \alpha_{2k} \\ \vdots \\ \alpha_{mk} \end{bmatrix} \neq \bar{0}_{m\times 1}$ because $\dim(\mathcal{R}(A)) = r$.

Thus each column of A is a linear combination of r non-zero column vectors in $K^{m\times 1}$. Then $\dim(\mathcal{C}(A)) \leq r$, i.e., the column rank of A is less than or equal to r. Therefore, $\dim(\mathcal{C}(A)) \leq \dim(\mathcal{R}(A))$. Similarly, $\dim(\mathcal{R}(A)) \leq \dim(\mathcal{C}(A))$. Therefore, $\dim(\mathcal{C}(A)) = \dim(\mathcal{R}(A))$. □

The row operations on a matrix over a skew field can also be done the same procedure as in the case of a matrix over a field. Consequently, an elementary row matrix is also defined to be a matrix derived by applying row operation exactly one time on an identity matrix. Then the reduced-row matrix, derived by applying row operations on a matrix, can also be written as the product of elementary row matrices and the original matrix.

Theorem 1.3. *For an arbitrary $m \times n$ matrix A over a skew field K, the rank of A is equal to the number of all nonzero rows of the reduced-row echelon matrix of A.*

Proof. Let A be an $m \times n$ matrix over a skew field K, and A_{RR} denote the reduced-row echelon matrix of A such that $A_{RR} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_r \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

where $R_i \neq \bar{0}_{m\times n}$ for all $i = 1, 2, \ldots, r$ and $0 < r \leq \min\{m, n\}$. We will show that $\{R_1, R_2, \ldots, R_r\}$ is linearly independent. Suppose $\{R_1, R_2, \ldots, R_r\}$ is not linearly independent, i.e., there exists $j \in \{1, 2, \ldots, r\}$ such that R_j is a linear combination of the remaining rows.

Let

$$R_j = \alpha_1 R_1 + \alpha_2 R_2 + \cdots + \alpha_{j-1} R_{j-1} + \alpha_{j+1} R_{j+1} + \cdots + \alpha_r R_r. \qquad (1.3.1)$$

Suppose $R_j = (0, 0, \ldots, 0, b, \ldots)$ where $b \neq 0$ is the leading entry and b is an element in a column k. Since R_j is the row of A_{RR}. Then other entries in the column k, not a leading entries, are all zero. Then the element in the column k of (1.3.1) becomes

$$b = \alpha_1 0 + \alpha_2 0 + \cdots + \alpha_{j-1} 0 + \alpha_{j+1} 0 + \cdots + \alpha_r 0 = 0.$$

This result contradicts to the assumption that $b \neq 0$. Therefore, $R_1, R_2, \ldots, R_r$ is linearly independent. Hence $\{R_1, R_2, \ldots, R_r\}$ becomes a basis of the row space of A. Then $\dim(\mathcal{R}(A)) = r$, i.e., $r(A) = r$ is equal to the number of all nonzero rows of the reduced-row echelon matrix of A. □

Next, we will state the degree theorem for the sum of two subspaces of a vector space over a skew field. The proof of this theorem can be done by the same method as for a vector space over a field. We then omit its proof.

Theorem 1.4. *Let V be a vector space over a skew field K, U and W subspaces of V. Define the sum $U + W = \{u + w \mid u \in U, w \in W\}$. Then $U + W$ is the smallest subspace of V containing both U and W such that*

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

2 Rank of an Upper Triangular Block Matrix Related to Row Spaces and Column Spaces

In this section, we will derive rank equations for a block matrix. For $A \in K^{m\times n}$ and $B \in K^{m\times t}$, we denote $\mathcal{C}\left(\begin{bmatrix} A & B \end{bmatrix}\right)$ a subspace of $K^{m\times 1}$ spanned by the columns of A and B, and $\mathcal{C}(A)+\mathcal{C}(B)$ the smallest subspace of $K^{m\times 1}$ containing both the column space of A and the column space of B. On the other hand, if $A \in K^{m\times n}$ and $B \in K^{s\times n}$, we denote $\mathcal{R}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right)$ a subspace of $K^{1\times n}$ spanned by the rows of A and B, and $\mathcal{R}(A) + \mathcal{R}(B)$ the smallest subspace of $K^{1\times n}$ containing both the row space of A and the row space of B.

Lemma 2.1. [4] *Let $A \in K^{s\times n}$ and $B \in K^{n\times m}$. Then $\mathcal{C}(AB) \subseteq \mathcal{C}(A)$ and $\mathcal{R}(AB) \subseteq \mathcal{R}(B)$. Moreover, $r(AB) \leq \min\{r(A), r(B)\}$.*

Lemma 2.2. *Let A and B be matrices over a skew field K.*

1. *If $A \in K^{m\times n}$ and $B \in K^{m\times t}$, then $\mathcal{C}\left(\begin{bmatrix} A & B \end{bmatrix}\right) = \mathcal{C}(A) + \mathcal{C}(B)$.*
   Moreover, $\dim\left(\mathcal{C}(A) \cap \mathcal{C}(B)\right) = r(A) + r(B) - r\left(\begin{bmatrix} A & B \end{bmatrix}\right)$.

2. *If* $A \in K^{m\times n}$ *and* $B \in K^{s\times n}$, *then* $\mathcal{R}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) = \mathcal{R}(A) + \mathcal{R}(B)$.

   Moreover, $\dim\left(\mathcal{R}(A) \cap \mathcal{R}(B)\right) = r(A) + r(B) - r\left(\begin{bmatrix} A \\ B \end{bmatrix}\right)$.

Proof. 1. Let $A \in K^{m\times n}$ and $B \in K^{m\times t}$. By the property of $\mathcal{C}(A) + \mathcal{C}(B)$, $\mathcal{C}(A) + \mathcal{C}(B) \subseteq \mathcal{C}\left(\begin{bmatrix} A & B \end{bmatrix}\right)$. Since all linear combinations of columns of A and B must also be in $\mathcal{C}(A) + \mathcal{C}(B)$, $\mathcal{C}\left(\begin{bmatrix} A & B \end{bmatrix}\right) \subseteq \mathcal{C}(A) + \mathcal{C}(B)$. By Theorem 1.4, $r\left(\begin{bmatrix} A & B \end{bmatrix}\right) = \dim(\mathcal{C}(A) + \mathcal{C}(B)) = r(A) + r(B) - \dim\left(\mathcal{C}(A) \cap \mathcal{C}(B)\right)$. That is, $\dim\left(\mathcal{C}(A) \cap \mathcal{C}(B)\right) = r(A) + r(B) - r\left(\begin{bmatrix} A & B \end{bmatrix}\right)$.
2. Let $A \in K^{m\times n}$ and $B \in K^{s\times n}$. Then $A^T \in K^{n\times m}$ and $B^T \in K^{n\times s}$. We apply the above result to have that

$$\mathcal{R}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) = \mathcal{C}\left(\begin{bmatrix} A^T & B^T \end{bmatrix}\right) = \mathcal{C}(A^T) + \mathcal{C}(B^T) = \mathcal{R}(A) + \mathcal{R}(B).$$

Similarly, $\dim\left(\mathcal{R}(A) \cap \mathcal{R}(B)\right) = r(A) + r(B) - r\left(\begin{bmatrix} A \\ B \end{bmatrix}\right)$. □

Lemma 2.3. *Let* A *and* B *be matrices over a skew field* K.

1. *If* $A \in K^{m\times n}$ *and* $B \in K^{m\times t}$, *then* $\mathcal{C}(A) \cap \mathcal{C}(B) \neq \{\bar{0}\}$ *if and only if there are column vectors* $\bar{\alpha}, \bar{\beta}$ *such that* $A\bar{\alpha} = B\bar{\beta} \neq \bar{0}$.

2. *If* $A \in K^{m\times n}$ *and* $B \in K^{s\times n}$, *then* $\mathcal{R}(A) \cap \mathcal{R}(B) \neq \{\bar{0}\}$ *if and only if there are row vectors* $\bar{\gamma}, \bar{\theta}$ *such that* $\bar{\gamma}A = \bar{\theta}B \neq \bar{0}$.

Proof. 1. Let $A \in K^{m\times n}$ and $B \in K^{m\times t}$ such that $A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}$ and $B = \begin{bmatrix} B_1 & B_2 & \cdots & B_t \end{bmatrix}$ where A_i is the i^{th} column matrix of A , for all $i = 1, 2, 3, \ldots, n$, and B_j is the j^{th} column matrices of B, for all $j = 1, 2, 3, \ldots, t$. ($\Rightarrow$) Assume $\mathcal{C}(A) \cap \mathcal{C}(B) \neq \{\bar{0}\}$. Let $v \neq \bar{0}$ be a column matrix such that $v \in \mathcal{C}(A) \cap \mathcal{C}(B)$. Then there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in K$ and $\beta_1, \beta_2, \ldots, \beta_t \in K$ such that $v = A_1\alpha_1 + A_2\alpha_2 + \cdots + A_n\alpha_n$ and $v = B_1\beta_1 + B_2\beta_2 + \cdots + B_t\beta_t$. We let $\bar{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$ and $\bar{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_t \end{bmatrix}$. We have that $A\bar{\alpha} = A_1\alpha_1 + A_2\alpha_2 + \cdots + A_n\alpha_n = v = B_1\beta_1 + B_2\beta_2 + \cdots + B_t\beta_t = B\bar{\beta}$. Then there exist the column vectors $\bar{\alpha}, \bar{\beta}$ such that $A\bar{\alpha} = B\bar{\beta} \neq \bar{0}$. ($\Leftarrow$) Assume that there exist column vectors $\bar{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$ and $\bar{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_t \end{bmatrix}$ such that

$A\bar{\alpha} = B\bar{\beta} \neq \bar{0}$. Then $\bar{0} \neq A_1\alpha_1 + A_2\alpha_2 + \cdots + A_n\alpha_n = A\bar{\alpha} = B\bar{\beta} = B_1\beta_1 + B_2\beta_2 + \cdots + B_t\beta_t$. Let $v = A_1\alpha_1 + A_2\alpha_2 + \cdots + A_n\alpha_n = B_1\beta_1 + B_2\beta_2 + \cdots + B_t\beta_t$. Then $v \neq \bar{0}$ and $v \in \mathcal{C}(A) \cap \mathcal{C}(B)$. Therefore, $\mathcal{C}(A) \cap \mathcal{C}(B) \neq \{\bar{0}\}$.

2. Let $A \in K^{m\times n}$ and $B \in K^{s\times n}$. Then $A^T \in K^{n\times m}$ and $B \in K^{n\times s}$. From the above result proved in 1., we have that

$$\begin{aligned}
&\mathcal{R}(A) \cap \mathcal{R}(B) = \mathcal{C}(A^T) \cap \mathcal{C}(B^T) \neq \{\bar{0}\} \\
\iff\ & \text{there are column vectors } \bar{\alpha}, \bar{\beta} \text{ such that } A^T\bar{\alpha} = B^T\bar{\beta} \neq \bar{0}. \\
\iff\ & \text{there are column vectors } \bar{\alpha}, \bar{\beta} \text{ such that } (\bar{\alpha}^T A)^T = (\bar{\beta}^T B)^T \neq \bar{0}. \\
\iff\ & \text{there are row vectors } \bar{\gamma} = \bar{\alpha}^T, \bar{\theta} = \bar{\beta}^T \text{ such that } \bar{\gamma}A = \bar{\theta}B \neq \bar{0}.
\end{aligned}$$

□

By Lemma 2.2 and Lemma 2.3, we derive the following theorem.

Theorem 2.4. *Let A and B be matrices over a skew field K.*

1. If $A \in K^{m\times n}$ and $B \in K^{m\times t}$, then the followings are equivalent:

- *(i)* $r\left(\begin{bmatrix} A & B \end{bmatrix}\right) = r(A) + r(B)$.
- *(ii)* $\mathcal{C}(A) \cap \mathcal{C}(B) = \{\bar{0}\}$.
- *(iii) There are no column vectors $\bar{\alpha}, \bar{\beta}$ such that $A\bar{\alpha} = B\bar{\beta} \neq \bar{0}$.*

2. If $A \in K^{m\times n}$ and $B \in K^{s\times n}$, then the followings are equivalent:

- *(iv)* $r\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) = r(A) + r(B)$.
- *(v)* $\mathcal{R}(A) \cap \mathcal{R}(B) = \{\bar{0}\}$.
- *(vi) There are no row vectors $\bar{\gamma}, \bar{\theta}$ such that $\bar{\gamma}A = \bar{\theta}B \neq \bar{0}$.*

We then also get an inequality for rank of a 2×2 block matrix.

Theorem 2.5. *Let $A \in K^{m\times n}$, $B \in K^{s\times t}$ and $C \in K^{m\times t}$.*

1. $r(A) + r(B) \leq r\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) \leq r(A) + r\left(\begin{bmatrix} C \\ B \end{bmatrix}\right)$.

2. $r(A) + r(B) \leq r\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) \leq r\left(\begin{bmatrix} A & C \end{bmatrix}\right) + r(B)$.

Proof. Let $A \in K^{n\times n},\ B \in K^{s\times t}$ and $C \in K^{m\times t}$.
It is shown in [4] that $r(A)+r(B) = r\left(\begin{bmatrix} A & 0 \\ 0 & B\end{bmatrix}\right) \le r\left(\begin{bmatrix} A & C \\ 0 & B\end{bmatrix}\right)$.
By Lemma 2.2, we have that $r\left(\begin{bmatrix} A & C \\ 0 & B\end{bmatrix}\right) \le r(A) + r\left(\begin{bmatrix} C \\ B\end{bmatrix}\right)$
and $r\left(\begin{bmatrix} A & C \\ 0 & B\end{bmatrix}\right) \le r\left(\begin{bmatrix} A & C\end{bmatrix}\right) + r(B)$. □

By Theorem 2.4 and Theorem 2.5, we derive the conditions for equalities of rank of a block matrix as follows.

Corollary 2.6. *Let $A \in K^{m\times n},\ B \in K^{s\times t}$ and $C \in K^{m\times t}$.*

1. *The followings are equivalent:*

   (i) $r\left(\begin{bmatrix} A & C \\ 0 & B\end{bmatrix}\right) = r(A) + r\left(\begin{bmatrix} C \\ B\end{bmatrix}\right)$.

   (ii) $\mathcal{C}\left(\begin{bmatrix} A \\ 0\end{bmatrix}\right) \cap \mathcal{C}\left(\begin{bmatrix} C \\ B\end{bmatrix}\right) = \left\{\begin{bmatrix} 0 \\ 0\end{bmatrix}\right\}$.

   (iii) There are no matrices X, Y, Z, W such that

$$\begin{bmatrix} A \\ 0\end{bmatrix}\begin{bmatrix} X & Y\end{bmatrix} = \begin{bmatrix} C \\ B\end{bmatrix}\begin{bmatrix} Z & W\end{bmatrix} \ne \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}.$$

2. *The followings are equivalent:*

   (iv) $r\left(\begin{bmatrix} A & C \\ 0 & B\end{bmatrix}\right) = r\left(\begin{bmatrix} A & C\end{bmatrix}\right) + r(B)$.

   (v) $\mathcal{R}\left(\begin{bmatrix} A & C\end{bmatrix}\right) \cap \mathcal{R}\left(\begin{bmatrix} 0 & B\end{bmatrix}\right) = \left\{\begin{bmatrix} 0 & 0\end{bmatrix}\right\}$.

   (vi) There are no matrix X, Y, Z, W such that

$$\begin{bmatrix} X \\ Y\end{bmatrix}\begin{bmatrix} A & C\end{bmatrix} = \begin{bmatrix} Z \\ W\end{bmatrix}\begin{bmatrix} 0 & B\end{bmatrix} \ne \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}.$$

In the last section, we will use a generalized inverse to find the necessary and sufficient conditions such that $r\left(\begin{bmatrix} A & C \\ 0 & B\end{bmatrix}\right) = r(A) + r(B)$.

3 Rank of an Upper Triangular Block Matrix and Generalized Inverses

It is common known that an inverse of a square matrix A is the unique matrix A^{-1} such that $A^{-1}A = AA^{-1} = I$ where I is the identity matrix. However, an inverse in this sense is of a different type from the usual inverse that we are familiar with. To be precise, in our work, an inverse is defined for a matrix of an arbitrary size (need not to be a square matrix) in the more generalized sense (see [1]). Most of the results in this section are similar results shown in [3] but for a block matrix over a skew field instead.

Definition 3.1. *Let $A \in K^{m\times n}$. We call B a generalized inverse or weak inverse or an inner inverse of A if $ABA = A$. We use A^{-} to represent an arbitrary generalized inverse of A.*

Definition 3.2. *Let $A \in K^{m\times n}$. If $U \in K^{n\times m}$ satisfies $UA = I_n$, then U is called a left inverse of A. Similarly, if $V \in K^{n\times m}$ satisfies $AV = I_m$, then V is called a right inverse of A.*

It is known (e.g. see [3]) that A has a left inverse if and only if $r(A) = n$, called *full column rank*, and A has a right inverse if and only if $r(A) = m$, called *full row rank*.

Theorem 3.3. [3] *Let $A \in K^{m\times n}$.*

(i) A has a left inverse if and only if $r(A) = \min\{m, n\}$

(ii) A has a right inverse if and only if $r(A) = \min\{m, n\}$

Lemma 3.4. *(Full rank decomposition) Let $A \in K^{m\times n}$ with $r(A) = r > 0$ where $r \leq \min\{m, n\}$. Then*

$$A = CR$$

where $C \in K^{m\times r}$ has a left-inverse and $R \in K^{r\times n}$ has a right inverse.

Proof. Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$. Denote $R_i = (a_{i1}, a_{i2}, \ldots, a_{in})$, for all $i = 1, 2, 3, \ldots, m$, the i^{th} row of A. Then there exist r rows of A forming a basis of the row space of A, without loss of generality, say $R_1,\ R_2, \ldots, R_r$. Then $R_{r+1},\ R_{r+2}, \ldots, R_m$ is a linear combination of $R_1,\ R_2, \ldots, R_r$.

case 1 $R_1, R_2, \ldots, R_r$ form a basis of $R(A)$.
We obtain $R_1 = R_1 + 0R_2 + \cdots + 0R_r, R_2 = 0R_1 + R_2 + \cdots + 0R_r, \ldots,$
$R_r = 0R_1 + 0R_2 + \cdots + R_r, R_{r+1} = \alpha_{(r+1)1}R_1 + \alpha_{(r+1)2}R_2 + \cdots + \alpha_{(r+1)r}R_r,$
$\ldots, R_m = \alpha_{m1}R_1 + \alpha_{m2}R_2 + \cdots + \alpha_{mr}R_r$
for some $\alpha_{ij} \in K$ where $i \in \{r+1, r+2, \ldots, m\},\ j \in \{1, 2, \ldots, r\}$.

Then $A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \alpha_{(r+1)1} & \alpha_{(r+1)2} & \cdots & \alpha_{(r+1)r} \\ \alpha_{(r+2)1} & \alpha_{(r+2)2} & \cdots & \alpha_{(r+2)r} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mr} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_r \end{bmatrix}$.

Therefore, A can be written as $A = CR$ where $C \in K^{m\times r}$ has a left inverse and $R \in K^{r\times n}$ has a right inverse.

case 2 $R_{i_1}, R_{i_2}, \ldots, R_{i_r}$ form a basis of $R(A)$ where $i_1 < i_2 < \ldots < i_r$ and $i_1, i_2, \ldots, i_r \in \{1, 2, \ldots, m\}$. We then rewrite $\{R_1, R_2, \ldots, R_m\}$ to be the ordered set $\{R_{i_1}, R_{i_2}, \ldots, R_{i_r}, \ldots, R_{i_m}\}$.

Let $\bar{A} = \begin{bmatrix} R_{i_1} \\ R_{i_2} \\ \vdots \\ R_{i_r} \\ \vdots \\ R_{i_m} \end{bmatrix}$. We have that A is row equivalent to $\bar{A}$. Then there are elementary matrices $e_1, e_2, \ldots, e_k$ such that $\bar{A} = e_k e_{k-1} \ldots e_1 A$. By case 1, there are matrices $\bar{C} \in K^{m\times r}$ and $\bar{R} \in K^{r\times n}$ such that $\bar{A} = \bar{C}\bar{R}$ where $\bar{C}$ has a left inverse and $\bar{R}$ has a right inverse. Then $A = e_1^{-1}e_2^{-1}\ldots e_k^{-1}\bar{A} = e_1^{-1}e_2^{-1}\ldots e_k^{-1}\bar{C}\bar{R}$. Choose $C = e_1^{-1}e_2^{-1}\ldots e_k^{-1}\bar{C}$. Then C also has a left inverse. Therefore, A can be written as $A = C\bar{R}$ where $C \in K^{m\times r}$ has a left inverse and $\bar{R} \in K^{r\times n}$ has a right inverse.

We will also show the other way to find a decomposition of A (implying that such decomposition is not unique).

Let $C_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$ for all $j = 1, 2, 3, \ldots, n$. Then there exist r columns of A forming a basis of the column space of A, without loss of generality, say

$C_1,\ C_2,\dots,C_r$. Hence $C_{r+1},\ C_{r+2},\dots,C_n$ is a linear combination of $C_1,\ C_2,\dots,C_r$. We obtain $C_1 = C_1 + C_2 0 + \cdots + C_r 0$,
$C_2 = C_1 0 + C_2 + \cdots + C_r 0, \dots, C_r = C_1 0 + C_2 0 + \cdots + C_r$,
$C_{r+1} = C_1\beta_{(r+1)1} + C_2\beta_{(r+1)2} + \cdots + C_r\beta_{(r+1)r}, \dots,$
$C_n = C_1\beta_{n1} + C_2\beta_{n2} + \cdots + C_r\beta_{nr}$
for some $\beta_{ij} \in K$ where $i \in \{r+1, r+2, \dots, n\}$ and $j \in \{1, 2, \dots, r\}$.
Then $A = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix}$

$$= \begin{bmatrix} C_1 & C_2 & \cdots & C_r \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & \beta_{(r+1)1} & \beta_{(r+2)1} & \cdots & \beta_{n1} \\ 0 & 1 & \cdots & 0 & \beta_{(r+1)2} & \beta_{(r+2)2} & \cdots & \beta_{n2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \beta_{(r+1)r} & \beta_{(r+2)r} & \cdots & \beta_{nr} \end{bmatrix}.$$

Therefore, A can be written as $A = CR$ where $C \in K^{m\times r}$ has a left-inverse and $R \in K^{r\times n}$ has a right inverse. □

Theorem 3.5. *(Existence of a generalized inverse) For $A \in K^{m\times n}$, there are nonsingular matrices $F \in K^{m\times m},\ G \in K^{n\times n},\ r(A) = r > 0$ where $r \leq \min\{m, n\}$ such that*

$$A = F \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} G$$

where some bordering zero matrices are absent if A has full rank.
Moreover, A has a generalized inverse $A^- = G^{-1} \begin{bmatrix} I_r & X \\ Y & Z \end{bmatrix} F^{-1}$ such that $X \in K^{r\times(n-r)},\ Y \in K^{(m-r)\times r}$ and $Z \in K^{(m-r)\times(n-r)}$.

Proof. Let $A \in K^{m\times n}$. By Lemma 3.4, there are a left invertible matrix $C \in K^{m\times r}$ and a right invertible matrix $R \in K^{r\times n}$ such that $A = CR$.
Let $F = \begin{bmatrix} C & C_0 \end{bmatrix}$ and $G = \begin{bmatrix} R \\ R_0 \end{bmatrix}$ where the matrices C_0 and R_0 are added, if necessary, so that F and G are still nonsingular matrices. Then $A = \begin{bmatrix} C & C_0 \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R \\ R_0 \end{bmatrix}$. It can be verified that for any X, Y, Z of appropriate dimensions $\begin{bmatrix} I_r & X \\ Y & Z \end{bmatrix}$, is a generalized inverse of $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. Then
$A\left(G^{-1} \begin{bmatrix} I_r & X \\ Y & Z \end{bmatrix} F^{-1}\right) A = A$. Thus $G^{-1} \begin{bmatrix} I_r & X \\ Y & Z \end{bmatrix} F^{-1}$ is a generalized inverse of A. □

Now, for any matrix A, we can always write A^- for any arbitrary generalized inverse of A. The following lemma is easily derived.

Lemma 3.6. *Let $A \in K^{m\times n}$. Then*

$$(I_n - A^-A)^2 = (I_n - A^-A) \quad \text{and} \quad (I_m - AA^-)^2 = (I_m - AA^-).$$

Theorem 3.7. *Let A, B be conformable matrices and for any choices of generalized inverse A^-,*
(i) $r\left(\begin{bmatrix}AB & (I_m - AA^-)C\end{bmatrix}\right) = r(AB) + r((I_m - AA^-)C)$,
(ii) $r\left(\begin{bmatrix}BA \\ C(I_n - A^-A)\end{bmatrix}\right) = r(BA) + r(C(I_n - A^-A))$.

Proof. (i) Let v be a column matrix such that $v \in \mathcal{C}(AB) \cap \mathcal{C}((I_m - AA^-)C)$. Then $AB\bar{\alpha} = v = (I_m - AA^-)C\bar{\beta}$ where $\bar{\alpha}$, $\bar{\beta}$ are column vectors. Since $(I_m - AA^-)^2 = (I_m - AA^-)$ (by Lemma 3.6), $(I_m - AA^-)AB\bar{\alpha} = \bar{0} = (I_m - AA^-)C\bar{\beta}$. Therefore, there are no column vectors $\bar{\alpha}$, $\bar{\beta}$ such that $AB\bar{\alpha} = (I_m - AA^-)C\bar{\beta} \neq \bar{0}$. Hence $r\begin{bmatrix}AB & (I - AA^-)C\end{bmatrix} = r(AB) + r((I_m - AA^-)C)$ by Theorem 2.4.
(ii) Let w be a row matrix such that $w \in \mathcal{R}(BA) \cap \mathcal{R}(C(I_n - A^-A))$. Then $\bar{\lambda}BA = w = \bar{\gamma}C(I_n - A^-A)$ where $\bar{\lambda}$, $\bar{\gamma}$ are row vectors. Since $(I_n - A^-A)^2 = (I_n - A^-A)$ (by Lemma 3.6), $\bar{\lambda}BA(I_n - A^-A) = \bar{0} = \bar{\gamma}C(I_n - A^-A)$. Therefore, there are no row vectors λ, γ such that $\bar{\lambda}BA(I_n - A^-A) = \bar{\gamma}C(I_n - A^-A) \neq \bar{0}$. By Theorem 2.4, $r\left(\begin{bmatrix}BA \\ C(I_n - A^-A)\end{bmatrix}\right) = r(BA) + r(C(I_n - A^-A))$. □

Theorem 3.8. *(Cancellation Rules). Let $C \in K^{m\times m}$ have a left inverse (full column rank) and $B \in K^{n\times n}$ have a right inverse (full row rank). Then for any matrix $A \in K^{m\times n}$,*

$$r(A) = r(CA) = r(AB).$$

Proof. Let $C \in K^{m\times m}$ and $B \in K^{n\times n}$ such that C has a left inverse (full column rank) and B has a right inverse (full row rank). We let L be a left inverse of C and R be a right inverse of B. Let $A \in K^{m\times n}$. By Proposition 2.1, $r(A) = r(LCA) \leq r(CA) \leq r(A)$, and $r(A) = r(ABR) \leq r(AB) \leq r(A)$. Therefore, $r(A) = r(CA) = r(AB)$. □

Theorem 3.9. *Let $A \in K^{m\times n}$, $B \in K^{m\times t}$ and $C \in K^{s\times n}$ and for any choices of their generalized inverses A^-, B^-, C^-,*
(i) $r\left(\begin{bmatrix}A & B\end{bmatrix}\right) = r(A) + r\left((I_m - AA^-)B\right) = r\left((I_m - BB^-)A\right) + r(B)$,
(ii) $r\left(\begin{bmatrix}A \\ C\end{bmatrix}\right) = r(A) + r(C(I_n - A^-A)) = r(A(I_n - C^-C)) + r(C)$.

Proof. (i) Since $\begin{bmatrix} I_n & A^-B \\ 0 & I_t \end{bmatrix}$ is a right inverse of $\begin{bmatrix} I_n & -A^-B \\ 0 & I_t \end{bmatrix}$ and $\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} I_n & -A^-B \\ 0 & I_t \end{bmatrix}$, by Theorem 3.8 and Theorem 3.7(i),

$$r\left(\begin{bmatrix} A & B \end{bmatrix}\right) = r\left(\begin{bmatrix} A & (I_m - AA^-)B \end{bmatrix}\right) = r(A) + r((I_m - AA^-)B).$$

(ii) Since $\begin{bmatrix} I_m & 0 \\ CA^- & I_s \end{bmatrix}$ is a left inverse of $\begin{bmatrix} I_m & 0 \\ -CA^- & I_s \end{bmatrix}$ and $\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ CA^- & I_s \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix}$, by Theorem 3.8 and Theorem 3.7(ii), we have

$$r\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) = r\left(\begin{bmatrix} A \\ -CA^-A + C \end{bmatrix}\right) = r(A) + r(C(I_n - A^-A)).$$

□

Theorem 3.10. *Let* $A \in M^{m\times n}$, $B \in M^{s\times t}$ *and* $C \in M^{m\times t}$. *Then* $r\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) = r(A) + r(B)$ *if and only if* $(I_m - AA^-)C(I_t - B^-B) = \bar{0}_{m\times t}$.

Proof. Since $\begin{bmatrix} 0 \\ B^- \end{bmatrix}$ is a generalized inverse of $\begin{bmatrix} 0 & B \end{bmatrix}$, we have

$$I_{n+t} - \begin{bmatrix} 0 & B \end{bmatrix}^- \begin{bmatrix} 0 & B \end{bmatrix} = I_{n+t} - \begin{bmatrix} 0 \\ B^- \end{bmatrix} \begin{bmatrix} 0 & B \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_t - B^-B \end{bmatrix},$$

and

$$\begin{aligned} \begin{bmatrix} A & C \end{bmatrix} \left(I_{n+t} - \left(\begin{bmatrix} 0 & B \end{bmatrix}^- \begin{bmatrix} 0 & B \end{bmatrix}\right)\right) &= \begin{bmatrix} A & C \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & I_t - B^-B \end{bmatrix} \\ &= \begin{bmatrix} A & C(I_t - B^-B) \end{bmatrix}. \end{aligned}$$

That is, by Theorem 3.9,

$$\begin{aligned} r\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) &= r\left(\begin{bmatrix} 0 & B \end{bmatrix}\right) + r\left(\begin{bmatrix} A & C \end{bmatrix}\left(I_{n+t} - \begin{bmatrix} 0 & B \end{bmatrix}^- \begin{bmatrix} 0 & B \end{bmatrix}\right)\right) \\ &= r(B) + r\left(\begin{bmatrix} A & C(I_t - B^-B) \end{bmatrix}\right) \\ &= r(B) + r(A) + r\left((I_m - AA^-)(C - CB^-B)\right) \\ &= r(A) + r(B) + r\left(C - CB^-B - AA^-C + AA^-CB^-B\right) \\ &= r(A) + r(B) + r\left((C - AA^-C)(I_t - B^-B)\right) \\ &= r(A) + r(B) + r\left((I_m - AA^-)C(I_t - B^-B)\right). \end{aligned}$$

Thus $r\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) = r(A) + r(B)$ if and only if $(I_m - AA^-)C(I_t - B^-B) = \bar{0}_{m\times t}$. □

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