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On Symmetric Bi-Multipliers of Lattice Implication Algebras

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Abstract

In this paper, we introduce the notion of symmetric bi-multiplier of lattice implication algebra and investigated some related properties. Also, we prove that if D is a symmetric bi-multiplier of L, then d_a is an isotone map of L.

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1 Introduction

In order to research a logical system whose propositional value is given in a lattice. Y. Xu [7] proposed the concept of lattice implication algebras, and some researchers have studied their properties and the corresponding logic systems. Also, in [8], Y. Xu and K. Y. Qin discussed the properties lattice H implication algebras, and gave some equivalent conditions about lattice H implication algebras. Y. Xu and K. Y. Qin [9] introduced the notion of filters in a lattice implication, and investigated their properties. In this paper, we introduce the notion of symmetric bi-multiplier of lattice implication algebra and investigated some related properties. Also, we prove that if D is a symmetric bi-multiplier of L, then d_a is an isotone map of L.

2 Preliminaries

A *lattice implication algebra* is an algebra $(L; \wedge, \vee, ', \rightarrow, 0, 1)$ of type $(2, 2, 1, 2, 0, 0)$, where $(L; \wedge, \vee, 0, 1)$ is a bounded lattice, " $'$ " is an order-reversing involution and " $\rightarrow$ " is a binary operation, satisfying the following axioms:

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$ for any $x, y, z \in L$,

(I2) $x \rightarrow x = 1$ for any $x \in L$,

(I3) $x \rightarrow y = y' \rightarrow x'$ for any $x, y \in L$,

(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$ for any $x, y \in L$,

(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$ for any $x, y \in L$,

(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$ for any $x, y, z \in L$,

(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$ for any $x, y, z \in L$.

If L satisfies conditions (I1) – (I5), we say that L is a *quasi lattice implication algebra*. A lattice implication algebra L is called a *lattice H implication algebra* if it satisfies $x \vee y \vee ((x \wedge y) \rightarrow z) = 1$ for all $x, y, z \in L$.

In the sequel the binary operation " $\rightarrow$ " will be denoted by juxtaposition. We can define a partial ordering " $\leq$ " on a lattice implication algebra L by $x \leq y$ if and only if $x \rightarrow y = 1$.

In a lattice implication algebra L, the following hold (see [7]):

(u1) $0 \rightarrow x = 1$, $1 \rightarrow x = x$ and $x \rightarrow 1 = 1$ for any $x \in L$,

(u2) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$ for any $x, y, z \in L$,

(u3) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$ for any $x, y, z \in L$,

(u4) $x' = x \rightarrow 0$ for any $x \in L$,

(u5) $x \vee y = (x \rightarrow y) \rightarrow y$ for any $x, y \in L$,

(u6) $((y \rightarrow x) \rightarrow y')' = x \wedge y = ((x \rightarrow y) \rightarrow x')'$ for any $x, y \in L$,

(u7) $x \leq (x \rightarrow y) \rightarrow y$ for any $x, y \in L$.

In a lattice H implication algebra L, the following hold:

(u8) $x \rightarrow (x \rightarrow y) = x \rightarrow y$ for any $x, y \in L$,

(u9) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$ for any $x, y, z \in L$.

A subset F of a lattice implication algebra L is called a *filter* of L it it satisfies:

(F1) $1 \in F$,

(F2) $x \in F$ and $x \to y \in F$ imply $y \in F$, for all $x, y \in L$.

Definition 2.1 *Let L be a lattice implication algebra. A mapping $D : L \times L \to L$ is called symmetric if $D(x, y) = D(y, x)$ holds for all $x, y \in L$.*

Definition 2.2 *Let L be a lattice implication algebra and $x \in L$. A mapping $d(x) = D(x, x)$ is called trace of D, where $D : L \times L \to L$ is a symmetric mapping on L.*

3 Symmetric bi-multipliers of lattice implication algebras

In what follows, let L denote a lattice implication algebra unless otherwise specified.

Definition 3.1 *Let L be a lattice implication algebra. A symmetric map $D : L \times L \to L$ is called a symmetric bi-multiplier of L if the following condition hold:*

$$D(x \vee y, z) = x \vee D(y, z)$$

for all $x, y, z \in L$.

Example 3.2 *Let $L := \{0, a, b, 1\}$ be a set with the Cayley table.*

x	x'
0	1
a	b
b	a
1	0

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

For any $x \in L$, we have $x' = x \to 0$. The operations $\wedge$ and $\vee$ on L are defined as follows:

$$x \vee y = (x \to y) \to y, \qquad x \wedge y = ((x' \to y') \to y')'.$$

Then $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra. Define a map $D : L \times L \to L$ by

$$D(x, y) = \begin{cases} a & \textit{if } (x, y) = (0, 0) \\ b & \textit{if } (x, y) = (0, a) \text{ or } (x, y) = (a, 0) \\ 1, & \textit{otherwise} \end{cases}$$

It is easy to check that D is a symmetric bi-multiplier of L.

Proposition 3.3 *Let D be a symmetric bi-multiplier of L. Then $D(1,1) = 1$.*

Proof. Let D be a symmetric bi-multiplier of L. Then we have

$$\begin{aligned} D(1,1) &= D(1 \vee 1, 1) \\ &= 1 \vee D(1,1) = 1 \end{aligned}$$

Proposition 3.4 *Let D be a symmetric bi-multiplier of L. Then $D(1,x) = D(x,1) = 1$ for all $x \in L$.*

Proof. Let D be a symmetric bi-multiplier of L. Then we have

$$\begin{aligned} D(1,x) &= D(1 \vee 1, x) \\ &= 1 \vee D(1,x) = 1 \end{aligned}$$

for every $x \in L$. Similarly, $D(x,1) = 1$ for every $x \in L$.

Proposition 3.5 *Let D be a symmetric bi-multiplier of L. If d is a trace of D, then the following conditions hold:*
(1) *$D(x,y) = x \vee D(x,y)$ for all $x, y \in L$.*
(2) *$d(1) = 1$.*

Proof. (1) Let D be a symmetric bi-multiplier of L. Then we have

$$\begin{aligned} D(x,y) &= D(x \vee x, y) \\ &= x \vee D(x,y) \end{aligned}$$

for all $x, y \in L$.
(2) It is clear from (1).

Proposition 3.6 *Let D be a symmetric bi-multiplier of L. If d is a trace of D, then $d(x) = d(x) \vee x$ for all $x \in L$.*

Proof. Let d be a trace of symmetric bi-multiplier D of L. Then we have

$$\begin{aligned} d(x) &= D(x,x) = D(x \vee x, x) \\ &= x \vee D(x,x) = x \vee d(x) \end{aligned}$$

for all $x \in L$. This completes the proof.

Corollary 3.7 *Let D be a symmetric bi-multiplier of L. If d is a trace of D, then $x \leq d(x)$ for all $x \in L$.*

Proposition 3.8 *Let D be a symmetric bi-multiplier of L. Then $D(x,y) \geq x$ and $D(x,y) \geq y$ for all $x, y \in L$.*

Proof. Let D be a symmetric bi-multiplier of L. Then we have

$$\begin{aligned} D(x,y) &= D(x \vee x, y) = x \vee D(x,y) \\ &= (x \to D(x,y)) \to D(x,y) \geq x \end{aligned}$$

for all $x, y \in L$ by (u7). Similarly, we have $y \leq D(x,y)$ for all $x, y \in L$. This completes the proof.

Definition 3.9 *Let D be a symmetric bi-multiplier of L. If $x \leq w$ implies $D(x,y) \leq D(w,y)$, D is called an isotone symmetric bi-multiplier of L.*

Theorem 3.10 *Let L be a lattice implication algebra and let D be a symmetric bi-multiplier of L. Then D is an isotone map of L.*

Proof. Let $x, y \in L$ be such that $x \leq y$ and D be a symmetric bi-multiplier of L. Then

$$\begin{aligned} D(y,z) &= D((x \to y) \to y, z) = D(x \vee y, z) \\ &= D(y \vee x, z) = y \vee D(x,z) \\ &= (y \to D(x,z)) \to D(x,z) \\ &= (D(x,z) \to y) \to y \geq D(x,z) \end{aligned}$$

for all $z \in L$. This implies that D is an isotone map of L by (u7).

Let L be a lattice implication algebra and let D be a symmetric bi-multiplier of L. For a fixed element $a \in L$, define a map $d_a : L \to L$ by $d_a(x) = D(x,a)$ for all $x \in L$.

Proposition 3.11 *Let L be a lattice implication algebra and let D be a symmetric bi-multiplier of L. Then the following conditions hold:*

(1) *$d_a(x) = d_a(x) \vee x$ for every $x \in L$.*
(2) *$d_a(x \vee y) = x \vee d_a(y)$ for every $x, y \in L$.*
(3) *If $x \leq y$, then $d_a(x \vee y) = d_a(y) \vee y$ for $x, y \in L$.*

Proof. (1) For every $x \in L$, we have

$$\begin{aligned} d_a(x) &= D(x,a) = D(x \vee x, a) \\ &= x \vee D(x,a) = x \vee d_a(x) \end{aligned}$$

(2) For every $x, y \in L$, we have

$$\begin{aligned} d_a(x \vee y) &= D(x \vee y, a) \\ &= x \vee D(y,a) = x \vee d_a(y) \end{aligned}$$

(3) Let $x, y \in L$ be such that $x \leq y$. Then $x \to y = 1$. Hence

$$\begin{aligned} d_a(x \vee y) &= D(x \vee y, a) \\ &= D((x \to y) \to y, a) = D(y,a) = d_a(y) \end{aligned}$$

Proposition 3.12 *Let L be a lattice implication algebra and let D be a symmetric bi-multiplier of L. Then $d_a(1) = 1$.*

Proof. Let L be a lattice implication algebra and let D be a symmetric bi-multiplier of L.

$$\begin{aligned} d_a(1) &= D(1, a) = D(1 \vee 1, a) \\ &= 1 \vee D(1, a) = 1 \vee 1 = 1. \end{aligned}$$

This completes the proof.

Theorem 3.13 *Let L be a lattice implication algebra and let D be a symmetric bi-multiplier of L. Then d_a is an isotone map of L.*

Proof. Let $x, y \in L$ be such that $x \leq y$ and $z \in L$. Then

$$\begin{aligned} d_a(y) &= D(y, a) = D((x \to y) \to y, a) \\ &= D(x \vee y, a) = D(y \vee x, a) \\ &= y \vee D(x, a) = (y \to d_a(x)) \to d_a(x) \\ &= (d_a(x) \to y) \to y \geq d_a(x). \end{aligned}$$

This implies that d_a is an isotone map of L by (u7).

Let L be a lattice implication algebra and let D be a symmetric bi-multiplier of L. For a fixed element $a \in L$, define a set $Fix_a(L)$ by

$$Fix_a(L) = \{x \in L \mid D(x, a) = x\}.$$

Proposition 3.14 *Let L be a lattice implication algebra and let D be a symmetric bi-multiplier of L. If $x \in L$ and $y \in Fix_a(L)$, then $x \vee y \in Fix_a(L)$.*

Proof. Let $x \in L$ and $y \in Fix_a(L)$. Then we obtain

$$D(x \vee y, a) = x \vee D(y, a) = x \vee y.$$

This completes the proof.

Proposition 3.15 *Let L be a lattice implication algebra and let D be a symmetric bi-multiplier of L. Then $x \leq y$ and $x \in Fix_a(L)$ implies $y \in Fix_a(L)$.*

Proof. Let x, y be such that $x \leq y$ and $x \in Fix_a(L)$. Then

$$\begin{aligned} D(y, a) &= D((x \to y) \to y, a) \\ &= D(x \vee y, a) = D(y \vee x, a) \\ &= y \vee D(x, a) = y \vee x \\ &= x \vee y = (x \to y) \to y \\ &= 1 \to y = y. \end{aligned}$$

This completes the proof.

Let D be a symmetric bi-multiplier of L and let d be a trace of D. Define a set $Kerd$ by

$$Kerd = \{x \in L \mid D(x, x) = d(x) = 1\}.$$

Proposition 3.16 *Let L be a lattice implication algebra and let D be a symmetric bi-multiplier of L. If $x \in L$ and $y \in Kerd$, then $x \vee y \in Kerd$.*

Proof. Let $x \in L$ and $y \in Kerd$. Then we obtain $d(y) = 1$. Hence

$$\begin{aligned} d(x \vee y) &= D(x \vee y, x \vee y) \\ &= x \vee D(x \vee y, y) = x \vee (x \vee D(y, y)) \\ &= x \vee (x \vee 1) = 1. \end{aligned}$$

Therefore, $x \vee y \in Kerd$. This completes the proof.

Proposition 3.17 *Let L be a lattice implication algebra and let D be a symmetric bi-multiplier of L. If $x \leq y$ and $x \in Kerd$, then $y \in Kerd$.*

Proof. Let $x \in Kerd$ and $x \leq y$. Then

$$\begin{aligned} d(y) &= D(y, y) = D((x \to y) \to y, (x \to y) \to y) \\ &= D(x \vee y, x \vee y) = D(y \vee x, y \vee x) \\ &= y \vee D(x, y \vee x) = y \vee D(y \vee x, x) \\ &= y \vee (y \vee D(x, x)) = y \vee (y \vee d(x)) \\ &= y \vee (y \vee 1) = 1 \end{aligned}$$

Therefore, this implies $y \in Kerd$. This completes the proof.

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