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New Results for the Fibonacci Sequence Using Binet's Formula

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Abstract

Let $k \geq 1$ and $h = 0, \ldots, k-1$. In this note we study the monotonicity of the sequences $(F_{kn+h})^{1/n}$, where F_n denotes the n-th Fibonacci number. In particular, we prove that the sequences $(F_{2n})^{1/n}$ and $(F_{2n+1})^{1/n}$ are strictly increasing for $n \geq 1$. We also obtain asymptotic formulae for $\sum_{k=1}^{n} \log F_k$, $\prod_{k=1}^{n} F_k$ and prove some limits.

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1 Introduction and Preliminary Notes

In 1202, the Italian mathematician Leonardo Fibonacci (1175-1251), through a problem (rabbit problem) in his book *Liber Abaci*, he introduced the following sequence of numbers:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

This sequence is called the *Fibonacci sequence*, and its terms are known as *Fibonacci numbers*. The Fibonacci sequence has a simple rule. In fact, starting with 0 and 1, every next number is found by adding up the two numbers before it. In mathematical terms, if F_n be the n-th Fibonacci number, then

$$F_n = F_{n-1} + F_{n-2}.$$

with $F_0 = 0$ and $F_1 = 1$.

There are many methods and explicit formulas to finding the n-th Fibonacci number. For example, the well-known *Binet's formula* (discovered by the French mathematician Jacques Philippe Marie Binet (1786-1856) in 1843) states that:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

The Binet's formula can also be written as

$$F_n = \frac{\varphi^n - (-\frac{1}{\varphi})^n}{\sqrt{5}}, \tag{1}$$

where $\varphi = \frac{1+\sqrt{5}}{2}(\approx 1.6180339887...)$, is the *golden ratio.*

There are many papers and books on the Fibonacci numbers. See, for example, [1]. In this article, we are interested in studying the monotonicity of sequences consisting of Fibonacci numbers and also in asymptotic formulae and limits where the Fibonacci Numbers appear.

2 Main Results

Lemma 2.1 *The following limit holds*

$$\lim_{x\longrightarrow 0} \frac{\log(1+x)}{x} = 1.$$

Proof. Use L' Hospital's rule. The lemma is proved.

Theorem 2.2 *Let $k \geq 1$ and $h = 0, \ldots, k-1$. If either $h = 0$ or $h = 1$ then the sequence $(F_{kn+h})^{\frac{1}{n}}$ is strictly increasing from a certain value of n and its limit is φ^k. On the other hand, if $h \geq 2$ then the sequence $(F_{kn+h})^{\frac{1}{n}}$ is strictly decreasing from a certain value of n and its limit is φ^k.*

Proof. We have (Binet's formula)

$$\begin{aligned}
(F_{kn+h})^{\frac{1}{n}} &= \left(\frac{\varphi^{kn+h}-\left(-\frac{1}{\varphi}\right)^{kn+h}}{\sqrt{5}}\right)^{\frac{1}{n}} \\
&= \varphi^k\left(\frac{\varphi^h}{\sqrt{5}}\right)^{\frac{1}{n}}\left(1+\frac{(-1)^{kn+h+1}}{\varphi^{2h}}\frac{1}{(\varphi^{2k})^n}\right)^{\frac{1}{n}} = \varphi^k a^{\frac{1}{n}}\left(1+\frac{c}{b^n}\right)^{\frac{1}{n}} \\
&= \exp\left(k\log\varphi+\frac{1}{n}\log a+\frac{1}{n}\log\left(1+\frac{c}{b^n}\right)\right) \\
&= \exp\left(k\log\varphi+\frac{1}{n}\log a+\frac{1}{n}\frac{f(n)c}{b^n}\right), \qquad (2)
\end{aligned}$$

where $a=\frac{\varphi^h}{\sqrt{5}}$, $c=\frac{(-1)^{kn+h+1}}{\varphi^{2h}}=O(1)$, $b=\varphi^{2k}>1$ and $f(n)\to 1$ by Lemma 2.1. Note that an immediate consequence of the second line of equation (2) is the limit of the sequence is φ^k.

Let us consider the sequence (see the last line of equation (2))

$$A_n = k\log\varphi+\frac{1}{n}\log a+\frac{1}{n}\frac{f(n)c}{b^n} = k\log\varphi+\frac{\log a}{n}+O\left(\frac{1}{nb^n}\right).$$

We have

$$A_{n+1}-A_n=\frac{-\log a}{n(n+1)}+O\left(\frac{1}{nb^n}\right)=\frac{1}{n(n+1)}\left(-\log a+O\left(\frac{n+1}{b^n}\right)\right).$$

Therefore, from a certain value of n the sign of $A_{n+1}-A_n$ is the sign of $-\log a = -\log\left(\frac{\varphi^h}{\sqrt{5}}\right)$. If either $h=0$ or $h=1$ then $-\log a>0$ and consequently the sequence A_n is strictly increasing (from a certain value of n). On the other hand, if $h\geq 2$ then $-\log a<0$ and consequently the sequence A_n is strictly decreasing (from a certain value of n). The theorem is proved.

In the case $k=2$ we can to prove that both sequences are strictly increasing from $n=1$.

Lemma 2.3 *Let $a>1.2841$ is a constant. Then for every $x>0$ we have*

$$2x\left(a^{2x}\log(a)+\frac{\log(a)}{a^{2x}}\right)>(a^{2x}-\frac{1}{a^{2x}})\log(a^{2x}-\frac{1}{a^{2x}}).$$

Proof. Let us consider the real function $h(x)=a^{2x}-\frac{1}{a^{2x}}$. The derivative of $h(x)$ is $h'(x)=\frac{2\log(a)(a^{4x}+1)}{a^{2x}}$ that is positive for $a>1$ and every real number x, which means that the function $h(x)$ (with $a>1$) is strictly increasing. In addition by calculation we have $h(x) > 1$ for $a \geq 1.2841$ and $x > 0$. Now let's continue the proof. We have

$$2xa^{2x}\log(a) = a^{2x}\log(a^{2x}) > a^{2x}\log(a^{2x} - \frac{1}{a^{2x}}) = a^{2x}\log(h(x)).$$

We have $\log(h(x)) > 0$ for $a \geq 1.2841$ and $x > 0$ (since $h(x) > 1$). Hence, we have

$$2xa^{2x}\log(a) - a^{2x}\log(a^{2x} - \frac{1}{a^{2x}}) > 0, \quad a \geq 1.2841, \forall x > 0. \tag{3}$$

Since $\frac{2x\log(a)}{a^{2x}}$ and $\frac{1}{a^{2x}}\log(a^{2x} - \frac{1}{a^{2x}})$ both are positive for $a > 1$ and $x > 0$, by adding these to the left side of inequality (3), we have

$$2xa^{2x}\log(a) - (a^{2x} - \frac{1}{a^{2x}})\log(a^{2x} - \frac{1}{a^{2x}}) + \frac{2x\log(a)}{a^{2x}} > 0, \quad \forall x > 0, \quad a \geq 1.2841.$$

Consequently

$$2x\left(a^{2x}\log(a) + \frac{\log(a)}{a^{2x}}\right) > (a^{2x} - \frac{1}{a^{2x}})\log(a^{2x} - \frac{1}{a^{2x}}), \quad \forall x > 0, \quad a \geq 1.2841.$$

The lemma is proved.

Theorem 2.4 *The sequence $(F_{2n})^{\frac{1}{n}}$ is strictly increasing for $n \geq 1$.*

Proof. Using Binet's formula we have

$$(F_{2n})^{\frac{1}{n}} = \left(\frac{\varphi^{2n} - (-\frac{1}{\varphi})^{2n}}{\sqrt{5}}\right)^{\frac{1}{n}} = \left(\frac{\varphi^{2n} - (\frac{1}{\varphi})^{2n}}{\sqrt{5}}\right)^{\frac{1}{n}}. \tag{4}$$

Let us consider the following continuous function corresponding to (4):

$$f(x) = \left(\frac{\varphi^{2x} - (\frac{1}{\varphi})^{2x}}{\sqrt{5}}\right)^{\frac{1}{x}}, \quad x > 0.$$

We obtain the derivative of $f(x)$ as follows:

$$f'(x) = \frac{\left(\varphi^{2x} - \frac{1}{\varphi^{2x}}\right)^{1/x}}{x^2(\sqrt{5})^{\frac{1}{x}}}\left(\frac{2x(\varphi^{2x}\log(\varphi) + \frac{\log(\varphi)}{\varphi^{2x}})}{(\varphi^{2x} - \frac{1}{\varphi^{2x}})} - \log\left(\varphi^{2x} - \frac{1}{\varphi^{2x}}\right)\right)$$

$$+\frac{\log(5)\left(\varphi^{2x} - \frac{1}{\varphi^{2x}}\right)^{1/x}}{2x^2(\sqrt{5})^{1/x}}.$$

Using Lemma 2.3 we know that $f'(x) > 0$ for $x > 0$. Since the function $f(x)$ has a positive derivative for $x > 0$, so $f(x)$ is strictly increasing for every $x > 0$, consequently the sequence $(F_{2n})^{\frac{1}{n}}$ is strictly increasing. The theorem is proved.

Lemma 2.5 *If $x > 0$ then the following inequality holds*

$$0 < \frac{\log(1+x)}{x} < 1.$$

Proof. The function $f(x) = x - \log(1+x)$ has positive derivative for $x > 0$ and $f(0) = 0$. The lemma is proved.

Theorem 2.6 *The sequence $(F_{2n+1})^{\frac{1}{n}}$ is strictly increasing for $n \geq 1$.*

Proof. If $k = 2$ and $h = 1$ equation (2) gives

$$(F_{2n+1})^{\frac{1}{n}} = \exp\left(2\log\varphi + \frac{1}{n}\log\left(\frac{\varphi}{\sqrt{5}}\right) + \frac{1}{n}f(n)\frac{1}{\varphi^2}\frac{1}{(\varphi^4)^n}\right) \quad (n \geq 1) \tag{5}$$

where, by Lemma 2.5, we have $0 < f(n) < 1$. Let us consider the sequence (see (5))

$$A_n = 2\log\varphi + \frac{1}{n}\log\left(\frac{\varphi}{\sqrt{5}}\right) + \frac{1}{n}f(n)\frac{1}{\varphi^2}\frac{1}{(\varphi^4)^n}.$$

We have

$$A_{n+1} - A_n = \frac{1}{n(n+1)}\left(-\log\left(\frac{\varphi}{\sqrt{5}}\right) + F(n)\right), \tag{6}$$

where

$$F(n) = f(n+1)\frac{1}{\varphi^2}\frac{n}{(\varphi^4)^{n+1}} - f(n)\frac{1}{\varphi^2}\frac{n+1}{(\varphi^4)^n},$$

and consequently

$$|F(n)| \leq \frac{1}{\varphi^2}\left(\frac{n}{(\varphi^4)^{n+1}} + \frac{n+1}{(\varphi^4)^n}\right) \leq \frac{2}{\varphi^2}\frac{n+1}{(\varphi^4)^n} \leq \frac{4}{\varphi^2}\frac{n}{(\varphi^4)^n} \leq \frac{4}{\varphi^6} \quad (n \geq 1) \tag{7}$$

Since the function $\frac{x}{(\varphi^4)^x}$ has negative derivative for $x \geq 1$.

On the other hand, we have the inequality $-\log\left(\frac{\varphi}{\sqrt{5}}\right) > \frac{4}{\varphi^6}$. Therefore equations (6) and (7) give $A_{n+1} - A_n > 0$ if $n \geq 1$. The theorem is proved.

Theorem 2.7 *The following asymptotic formulae and limits hold.*

$$\sum_{i=1}^{n} \log F_i = \frac{n^2}{2}\log\varphi + \frac{n}{2}\log\left(\frac{\varphi}{5}\right) + C + o(1), \tag{8}$$

where

$$C=\sum_{k=1}^{\infty}\log\left(1+\frac{(-1)^{k+1}}{\varphi^{2k}}\right). \tag{9}$$

$$\prod_{i=1}^{n}F_i\sim C'\varphi^{\frac{n^2}{2}}\left(\frac{\varphi}{5}\right)^{\frac{n}{2}}, \tag{10}$$

where

$$C'=\prod_{k=1}^{\infty}\left(1+\frac{(-1)^{k+1}}{\varphi^{2k}}\right). \tag{11}$$

$$\lim_{n\to\infty}(F_1F_2\cdots F_n)^{\frac{1}{n^2}}=\sqrt{\varphi}. \tag{12}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{F_1F_2\cdots F_n}}{F_n}=0. \tag{13}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{F_1F_2\cdots F_n}}{\sqrt{F_n}}=\sqrt{\frac{\varphi}{\sqrt{5}}}. \tag{14}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{\log F_3\log F_4\cdots\log F_n}}{\log F_n}=\frac{1}{e} \tag{15}$$

Let k be a positive integer, we have

$$\lim_{n\to\infty}\frac{\left((\log F_3)^{(3^k)}(\log F_4)^{(4^k)}\cdots(\log F_n)^{(n^k)}\right)^{\frac{k+1}{n^{k+1}}}}{\log F_n}=\frac{1}{e^{k+1}} \tag{16}$$

$$\lim_{n\to\infty}\left(\frac{\log F_{n+1}}{\log F_n}\right)^n=e$$

Proof. We have (Binet's formula)

$$F_n=\frac{1}{\sqrt{5}}\left(\varphi^n+(-1)^{n+1}\frac{1}{\varphi^n}\right)=\frac{1}{\sqrt{5}}\varphi^n\left(1+\frac{(-1)^{n+1}}{\varphi^{2n}}\right)\qquad(n\geq 1)$$

Therefore

$$F_n\sim\frac{1}{\sqrt{5}}\varphi^n, \tag{17}$$

and

$$\log F_n = n \log \varphi - \frac{1}{2} \log 5 + \log \left(1 + \frac{(-1)^{n+1}}{\varphi^{2n}} \right). \tag{18}$$

Hence

$$\sum_{i=1}^{n} \log F_i = \frac{n^2}{2} \log \varphi + \frac{n}{2} \log \left(\frac{\varphi}{5} \right) + \sum_{k=1}^{\infty} \log \left(1 + \frac{(-1)^{k+1}}{\varphi^{2k}} \right) + o(1).$$

Note that the series

$$\sum_{k=1}^{\infty} \log \left(1 + \frac{(-1)^{k+1}}{\varphi^{2k}} \right) = \sum_{k=1}^{\infty} f(k) \frac{(-1)^{k+1}}{\varphi^{2k}}$$

where $f(k) \to 1$ (by Lemma 2.1) converges absolutely. This proves equations (8) and (9). Equations (10) and (11) are an immediate consequence of equations (8) and (9). Limit (12) is an immediate consequence of equation (10). Limits (13) and (14) are an immediate consequence of equations (10) and (16).

Equation (17) gives

$$\log \log F_n = \log n + \log \log \varphi + o(1) \tag{19}$$

From the Stirling's formula $n! \sim \frac{\sqrt{2\pi} n^n \sqrt{n}}{e^n}$ we obtain

$$\sum_{i=1}^{n} \log i = n \log n - n + o(n) \tag{20}$$

Therefore (see (18) and (19))

$$\begin{aligned} & \log \left(\frac{\sqrt[n]{\log F_3 \log F_4 \cdots \log F_n}}{\log F_n} \right) = \frac{1}{n} \left(\log \log F_3 + \cdots + \log \log F_n \right) \\ - \quad & \log \log F_n = \frac{1}{n} \left(\sum_{i=1}^{n} \log i + n \log \log \varphi + o(n) \right) \\ - \quad & (\log n + \log \log \varphi + o(1)) = -1 + o(1) \end{aligned}$$

That is, equation (15).

The proof of equation (16) is the same as the proof of equation (15). Note that we have the well-known asymptotic formula

$$1^k + 2^k + \cdots + n^k \sim \frac{n^{k+1}}{k+1}$$

and since the function $x^k \log x$ is strictly increasing and integration by parts we have

$$\sum_{i=1}^{n} i^k \log i = \int_1^n x^k \log x \, dx + O\left(n^k \log n\right) = \frac{n^{k+1}}{k+1} \log n - \frac{1}{(k+1)^2} n^{k+1} + o\left(n^{k+1}\right)$$

The theorem is proved.

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References

[1] N. N. Vorobèv, *Fibonacci Numbers,* Springer, Birkhäuser Verlag, Basel, 2002. https://doi.org/10.1007/978-3-0348-8107-4

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