The Bruhat Ordering in Weyl Group of Type $D_n$

Yaoyao Jiao, Xigou Zhang and Hong Zhu

College of Mathematics & Information Science
JXNU, Nanchang, P.R.China, 330022

Abstract
In this paper, we deal with the Bruhat ordering in Coxeter groups of type $D_n$. At first, we consider the Bruhat ordering of $N'_n$. Then, we get some conclusions about Bruhat ordering between with $N'_n$, $W'_n$.

Keywords: Coxeter groups; Bruhat ordering; Partial ordering

1 Introduction

One of the most remarkable aspects of Coxeter groups from a combinatorial point of view is the crucial role that is played in their theory by a certain partial ordering structure. The study of Bruhat ordering has important theoretical significance. A lot of results have been generated. In 1977, Deodhar[2] got a criterion of Bruhat ordering on any Coxeter groups. Based on this, Bjorner and Brenti[3] got an improved criterion of Bruhat ordering on symmetric groups in 1996. In 1988, Jianyi Shi[5] gave some results about Bruhat ordering in Coxeter groups. In 2008, LiLi Fang[9] gave the Bruhat ordering in Coxeter groups of type $A_n$ and gave the directed combinatorics proof through the application of discriminant method. In 2017, Xiong and Zhang[10] gave the related conclusions about Bruhat ordering in Coxeter groups of type $B_n$. In this paper, we mainly consider the Bruhat ordering in Coxeter groups of type $D_n$ in some special cases. At first, we briefly introduce some related definitions of Bruhat ordering in Coxeter groups. Then, we get some conclusions about Bruhat ordering between with $N'_n$, $W'_n$.

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2 Prepare knowledge about the groups \( N'_n, W'_n \)

Let \((W, S)\) be a Coxeter system, i.e the pair \((W, S)\) consisting of a group \(W\) and a set of generators \(S\), subject only to relations of the form \( (st)^{m(s, t)} = 1 \) where \( m(s, s) = 1, m(s, t) = m(t, s) \geq 2 \) for \( s \neq t \) in \( S \). For any \( w \in W \) can be written as a product of elements in \( S \), say \( w = s_1s_2 \cdots s_r \). Define the length \( \ell(w) \) of \( w \) to be the smallest \( r \) for which such an expression exists, at the same time, \( s_1s_2 \cdots s_r \) is called a reduced expression of \( W \).

Let \( T = \{ wsw^{-1} \mid w \in W, s \in S \} \), write \( w' \rightarrow w \) if \( w = wt \) for some \( t \in T \) with \( \ell(w) > \ell(w') \). Then define \( w' \leq w \) if there is a sequence \( w' = w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_m = w \). We call " \( \leq \) " the Bruhat ordering. The subexpressions of a given reduced expression \( w = s_1s_2 \cdots s_r \) are products (not necessarily reduced, and possibly empty) of the form \( s_i \cdots s_q (1 \leq i < i_2 < \cdots < i_q \leq r) \).

Then \( w' \leq w \) if and only if \( w' \) can be obtained as a subexpression of the reduced expression \( w \). In this paper, we set \( \{1, q\} = \{1, 2, 3, \ldots, q\} \).

Let \( W_n = < t_0, s_1, \cdots, s_{n-1} > \) be Coxeter group of type \( B_n \). Then the following relations holds:
\[
\begin{align*}
t_0s_1t_0s_1 &= s_1t_0s_1t_0; t_0s_1 &= s_1t_0 \quad \text{for } i > 1; \\
s_is_{i+1}s_i &= s_{i+1}s_is_{i+1}; s_is_j &= s_js_i \quad \text{for } |i - j| > 1. \\
\end{align*}
\]

Let \( u = t_0s_1t_0 \), then \( u^2 = 1 \). We have \( u_1 = us_1 \) and \( u_i = s_is_{i-1}s_i \) for \( 2 \leq i \leq n-1 \).

Now set \( N'_n = < u_1, u_2, \ldots, u_{n-1} > \), \( A_{n-1} = < s_1, s_2, \ldots, s_{n-1} > \) and \( W'_n = < u, s_1, s_2, \ldots, s_{n-1} > \) is isomorphism to Coxeter group of type \( D_n \) and \( W'_n = N'_n \rtimes A_{n-1} \).

We have following relations:
\[
\begin{align*}
us_1 &= t_0s_1t_0s_1 = s_1t_0s_1t_0 = s_1u, us_2u &= s_2us_2, us_i &= s_iu \quad \text{for } i \geq 3. \\
\end{align*}
\]

Then we easily get the relations between \( u_i, s_j \) where \( i, j \in [1, n-1] \):
\[
\begin{align*}
u_is_j &= s_1u_1u_i \quad \text{if } j = 1, i \geq 2; \\
u_is_j &= s_iu_{i-1} \quad \text{if } j = i \geq 2; \\
u_is_j &= s_{i+1}u_{i+1} \quad \text{if } j = i + 1; \\
u_is_j &= s_ju_i \quad \text{otherwise.}
\end{align*}
\]

Next we discuss the Bruhat ordering in \( N'_n \) and in \( W'_n \).

3 The Bruhat ordering in \( N'_n \)

**Lemma 3.1.** Let \( u_i \) be as in above, then \( u_i \leq u_j \) if and only if \( i \leq j \).

**Proof.** By definition, we know \( u_i = u_iu_{i-1}u_i = s_is_{i-1} \cdots s_2u \quad s_1 \cdots s_{i-1}s_i \)
\[
\begin{align*}
u_j &= s_ju_{j-1}s_j = s_js_{j-1} \cdots s_2u \quad s_1 \cdots s_{j-1}s_j, \\
\ell(u_i) &= 2i, \ell(u_j) = 2j.
\end{align*}
\]

Obviously, we get this result. \( \square \)
Let \( w = w_1 \cdot w_2 \cdot \cdots \cdot w_r \) indicate \( w = w_1 w_2 \cdots w_r \) and \( l(w) = l(w_1) + l(w_2) + \cdots + l(w_r) \), \( w_i \in W \) with \( i \in [1,q] \). Let \( u_{i_1} \cdot u_{i_2} \cdot \cdots \cdot u_{i_p} \) and \( u_{i_1} \cdot u_{i_2} \cdot \cdots \cdot u_{i_q} \in N'_{n} \), we write \( u_{i_1} \cdot u_{i_2} \cdot \cdots \cdot u_{i_p} \leq N'_{n} u_{j_1} \cdot u_{j_2} \cdots \cdots \cdot u_{j_q} \in N'_{n} \) if \( p \leq q \), \( i_{p-k} \leq j_{q-k} \) for \( k \in [0,p-1] \). Then "\( \leq N'_{n} \)" is the Bruhat ordering in \( N'_{n} \) from the following lemma.

**Lemma 3.2.** Let \( w_1, w_2 \in N'_{n} \). Then \( w_1 \leq N'_{n} w_2 \) if and only if \( w_1 \leq w_2 \).

**Proof.** Let \( w_1 = u_{i_1} \cdot u_{i_2} \cdot \cdots \cdot u_{i_p} \) and \( w_2 = u_{j_1} \cdot u_{j_2} \cdots \cdots \cdot u_{j_q} \) as in above.

(\( \Rightarrow \)) It is obviously by definition and Lemma 3.1

(\( \Leftarrow \)) Conversely,

(i) If \( i_p = j_q \). By inductive hypothesis, we have:

\[
\begin{align*}
&u_{i_1} \cdot u_{i_2} \cdot \cdots \cdot u_{i_{p-1}} \leq u_{j_1} \cdot u_{j_2} \cdots \cdots \cdot u_{j_q}, \\
&u_{i_p} = u_{j_q} \text{ since } i_p = j_q.
\end{align*}
\]

Thus we have \( i_{p-k} \leq j_{q-k} \) for \( k \in [0,p-1] \).

(ii) If \( i_p < j_q \). We have:

\[
\begin{align*}
&u_{i_1} \cdot u_{i_2} \cdot \cdots \cdot u_{i_{p-1}} \leq u_{j_1} \cdot u_{j_2} \cdots \cdots \cdot u_{j_q} \\
&u_{i_p} = s_{i_p}s_{i_{p-1}} \cdots s_2 u_2 s_2 \cdots s_{i_{p-1}} s_i, \\
&u_{j_q} = s_{j_q}s_{j_{q-1}} \cdots s_2 u_2 s_2 \cdots s_{j_{q-1}} s_j.
\end{align*}
\]

Then \( u_{i_1} \cdot u_{i_2} \cdots \cdots \cdot u_{i_{p-1}} (s_{i_p}s_{i_{p-1}} \cdots s_2 u_2 s_2 \cdots s_{i_{p-1}} s_i) \leq u_{j_1} \cdot u_{j_2} \cdots \cdots \cdot u_{j_{q-1}} (s_{j_q}s_{j_{q-1}} \cdots s_{i_{p-1}} s_i) \).

Hence \( u_{i_1} \cdot u_{i_2} \cdots \cdots \cdot u_{i_{p-1}} \leq N' \cdot u_{j_1} \cdot u_{j_2} \cdots \cdots \cdot u_{j_{q-1}} \).

By inductive hypothesis, we have \( i_{p-k} \leq j_{q-k} \) for \( k \in [0,p-1] \). Thus, we have \( i_{p-k} \leq j_{q-k} \) for \( k \in [0,p-1] \). Hence, \( w_1 \leq N'_{n} w_2 \). \( \square \)

For any \( w \in W'_{n} \), there exist unique \( v \in A_{n-1} \) and \( w_1 \in N'_{n} \) such that \( w = vw_1 \) since \( W'_{n} \) is semi-direct product \( N'_{n} \) and \( A_{n-1} \). We denote \( w_1 = N'_{n} - w \)

**Lemma 3.3.** Let \( w = u_{t_1} \cdot t_r \) be reduced expression and \( u \) appears once, \( t_i \in \{ s_1, \cdots, s_{n-1} \} \) for \( i \in [1,r] \). Then we have:

(i) \( N' = w = u_{s_1} \cdot s_{i_2} \cdots s_{i_r} \) with \( s_{i_1}, s_{i_2} \cdots, s_{i_r} \neq s_2 \), then \( N' = u_{s_1} \cdot s_{i_2} \cdots s_{i_r} \).

(ii) \( N' = u_{s_1} \cdot s_{i_2} \cdots s_{i_r} \) with \( s_{i_1}, s_{i_2} \cdots, s_{i_r} \neq s_2 \), then \( N' = u_{s_1} \cdot s_{i_2} \cdots s_{i_r} \).

(iii) \( N' = u_{s_1} \cdot s_{i_2} \cdots s_{i_r} \) with \( k_1 = 1, 2, 3, \cdots, i_1-1, i_1+1, \cdots, n-1, \) \( i_1 \neq j \); \( N' = u_{s_1} \cdot s_{i_2} \cdots s_{i_r} \) with \( k_2 = 1, 2, 3, \cdots, n-1, \) \( i_1 \neq j \); and \( k_1, k_2, i, j \) are distinct.

**Proof.** (i) If \( w = u_{s_1} \cdot s_{i_2} \cdots s_{i_r} \) with \( s_{i_1}, s_{i_2} \cdots, s_{i_r} \neq s_2 \), we have:

\[
\begin{align*}
&us_{i_1} = u_{s_1} u_{s_3} u_{s_4} u_{s_6} u_{s_8} \cdots u_{s_{n-1}} u_{s_n}, \\
&us_{i_2} = u_{s_1} u_{s_3} u_{s_4} u_{s_6} u_{s_8} \cdots u_{s_{n-1}} u_{s_n}, \\
&us_{i_3} = u_{s_1} u_{s_3} u_{s_4} u_{s_6} u_{s_8} \cdots u_{s_{n-1}} u_{s_n}, \\
&\vdots \\
&us_{i_r} = u_{s_1} u_{s_3} u_{s_4} u_{s_6} u_{s_8} \cdots u_{s_{n-1}} u_{s_n}.
\end{align*}
\]

So \( N' = u_{s_1} u_{s_3} u_{s_4} u_{s_6} u_{s_8} \cdots u_{s_{n-1}} u_{s_n} = u_{s_1} \).

And \( u_{s_1} = u_{s_1} u_{s_3} u_{s_4} u_{s_6} u_{s_8} \cdots u_{s_{n-1}} u_{s_n} \). thus, \( N' = u_{s_1} \).
(ii) $u_{s_1} = u_1,$
$u_{s_1}s_2 = s_2u_2,$ $u_{s_2}s_1 = s_1s_2s_1u_1u_2,$ $u_{s_1}s_3 = s_3u_1 = u_{s_3}s_1,$
$u_{s_1}s_4 = s_4u_1 = u_{s_4}s_1,$ $u_{s_1}s_5 = s_5u_1 = u_{s_5}s_1,$ $\cdots,$ $u_{s_1}s_{n-1} = s_{n-1}u_1 = u_{s_{n-1}}s_1.$

Then $N' - u_{s_1} = u_1,$
$N' - u_{s_1}s_2 = u_2,$ $N' - u_{s_1}s_3 = u_1,$ $N' - u_{s_1}s_4 = u_1,$ $\cdots,$ $N' - u_{s_1}s_{n-1} = u_1;$$N' - u_{s_2}s_1 = u_1u_2,$ $N' - u_{s_3}s_1 = u_1,$ $N' - u_{s_4}s_1 = u_1,$ $\cdots,$ $N' - u_{s_{n-1}}s_1 = u_1;$$u_{s_2} = s_1s_2u_2,$
$u_{s_1}s_2 = s_2u_2,$ $u_{s_2}s_1 = s_1s_2s_1u_1u_2,$ $u_{s_3}s_2 = s_1s_2s_2u_2,$ $u_{s_2}s_3 = s_1s_2s_3u_3,$
$u_{s_4}s_2 = s_1s_4s_2u_2,$ $u_{s_2}s_4 = s_1s_2s_4u_2,$ $u_{s_5}s_2 = s_1s_5s_2u_2,$ $u_{s_2}s_5 = s_1s_2s_5u_2,$ $\cdots$$u_{s_{n-1}}s_2 = s_1s_{n-1}s_2u_2,$ $u_{s_2}s_{n-1} = s_1s_2s_{n-1}u_2;$$N' - u_{s_2} = u_2,$
$N' - u_{s_2}s_1 = u_1u_2,$ $N' - u_{s_3}s_2 = u_3,$ $N' - u_{s_4}s_2 = u_2,$ $\cdots,$ $N' - u_{s_{n-1}}s_2 = u_2;$$N' - u_{s_3}s_1 = u_1,$ $N' - u_{s_2}s_3 = u_2,$ $N' - u_{s_3}s_3 = u_3,$ $N' - u_{s_4}s_3 = u_1,$ $\cdots,$ $N' - u_{s_{n-1}}s_3 = u_1;$$u_{s_3} = s_1s_3u_1,$
$u_{s_1}s_3 = s_3u_1,$ $u_{s_3}s_1 = s_3u_1,$ $u_{s_3}s_3 = s_1s_2s_3u_3,$ $u_{s_3}s_2 = s_1s_3s_2u_2,$
$u_{s_3}s_3 = s_1s_4s_3u_1,$ $u_{s_3}s_4 = s_1s_3s_4u_1,$ $u_{s_5}s_3 = s_1s_5s_3u_1,$ $u_{s_3}s_5 = s_1s_3s_5u_1,$ $\cdots$$u_{s_{n-1}}s_3 = s_1s_{n-1}s_3u_1,$ $u_{s_3}s_{n-1} = s_1s_3s_{n-1}u_1;$$N' - u_{s_3} = u_1,$
$N' - u_{s_3}s_1 = u_1,$ $N' - u_{s_3}s_2 = u_2,$ $N' - u_{s_3}s_3 = u_1,$ $\cdots,$ $N' - u_{s_3}s_{n-1} = u_1;$$N' - u_{s_3}s_1 = u_1,$ $N' - u_{s_2}s_3 = u_2,$ $N' - u_{s_3}s_3 = u_3,$ $N' - u_{s_4}s_3 = u_1,$ $\cdots,$ $N' - u_{s_{n-1}}s_3 = u_1;$$u_{s_3} = s_1s_3u_1 (i \geq 4),$$u_{s_1}s_3 = s_1u_1,$ $u_{s_3}s_1 = s_1u_1,$ $u_{s_2}s_i = s_1s_2s_iu_2,$ $u_{s_3}s_2 = s_1s_3s_2u_2,$ $u_{s_3}s_i = s_1s_3s_iu_i,$ $u_{s_3}s_3 = s_1s_3s_3u_1,$ $u_{s_3}s_4 = s_1s_3s_4u_1,$ $\cdots,$ $u_{s_i-1}s_i = s_1s_{i-1}s_iu_1,$ $u_{s_i-1}s_i = s_1s_{i-1}s_iu_1,$ $u_{s_{i+1}}s_i = s_1s_{i+1}s_iu_1,$ $u_{s_{i+1}}s_{i+1} = s_1s_{i+1}s_{i+1}u_1,$ $\cdots,$ $u_{s_{n-1}}s_i = s_1s_{n-1}s_iu_1,$ $u_{s_{n-1}}s_{n-1} = s_1s_{n-1}s_{n-1}u_1.$

Then $N' - u_{s_{i}} = u_1 (i \geq 4),$$N' - u_{s_{i}}s_1 = u_1,$ $N' - u_{s_{i}}s_2 = u_2,$ $N' - u_{s_{i}}s_3 = u_1,$ $\cdots,$ $N' - u_{s_{i}}s_{i-1} = u_{s_{i}}s_{i+1} = u_1,$ $\cdots,$ $N' - u_{s_{i}}s_{n-1} = u_1,$
$N' - u_{s_{i}}s_i = u_1,$ $N' - u_{s_{i}}s_i = u_2,$ $N' - u_{s_{i}}s_i = u_1,$ $\cdots,$ $N' - u_{s_{i}}s_{i-1} = u_{s_{i}}s_{i+1} = u_1,$ $\cdots,$ $N' - u_{s_{i}}s_{n-1} = u_1;$$N' - u_{s_{i}}s_j \leq N' - u_{k_i}s_{k_j}, k_1 = 1,2,3,..,i-1,i+1,.,n-1, i \neq j;$$N' - u_{s_{i}}s_j \leq N' - u_{s_{j}}s_{k_j}, k_2 = 1,2,3,.,n-1 and k_2,i,j are distinct;}$
4 Some relations between the Bruhat ordering in $N'_n$ and in $W'_n$

Theorem 4.1. Let $w_1, w_2 \in W'_n$. Write $w_1 = us_is_{i+1} \cdots s_{i+p}, w_2 = us_is_{i+1} \cdots s_{i+q}, i + p, i + q \in [1, n-1], 1 \leq p \leq q$, then $N' - w_1 \leq N' - w_2$.

Proof. If we let $v_1 = s_is_{i+1}s_{i+2} \cdots s_{i+p}, v_2 = s_is_{i+1}s_{i+2} \cdots s_{i+q}, 1 \leq p \leq q$, $w_1 = vw_1, w_2 = vw_2 \in W'_n$. We have $w_1 \leq w_2$ since $v_1 \leq v_2$.

(i) If $i = 1$ or $2$, $N' - w_1 = u_{i+p}, N' - w_2 = u_{i+q}$, we have $N' - w_1 \leq N' - w_2$ since $i + p \leq i + q$ and $u_{i+p} \leq u_{i+q}$.

(ii) If $i \geq 3$, $N' - w_1 = N' - w_2 = u_1$, we have $N' - w_1 = N' - w_2$.

Hence, we have $N' - w_1 \leq N' - w_2$. □

Theorem 4.2. Let $w_1, w_2 \in W'_n$. Write $w_1 = us_is_{i+1}s_{i+1} \cdots s_{i+p}, w_2 = us_is_{i+1}s_{i+2} \cdots s_{i+q}, i + p, i + q \in [1, n-1], 1 \leq p \leq q$, then $N' - w_1 \leq N' - w_2$.

Proof. If we let $v_1 = s_is_{i+1}s_{i+1} \cdots s_{i+p}, v_2 = s_is_{i+1}s_{i+1} \cdots s_{i+q}, 1 \leq p \leq q$, $w_1 = vw_1, w_2 = vw_2 \in W'_n$. We have $w_1 \leq w_2$ since $v_1 \leq v_2$.

(i) If $i = 1$, $N' - w_1 = u_1 = N' - w_2$, we have $N' - w_1 = N' - w_2$.

(ii) If $i = 2$ or $3$, $N' - w_1 = u_1u_{i+p}, N' - w_2 = u_1u_{i+q}$, we have $N' - w_1 \leq N' - w_2$ since $i + p \leq i + q$ and $u_{i+p} \leq u_{i+q}$.

(iii) If $i \geq 4$, $N' - w_1 = u_1u_2 = N' - w_2$, we have $N' - w_1 = N' - w_2$.

Therefore, we have $N' - w_1 \leq N' - w_2$. □

Theorem 4.3. Let $w_1, w_2 \in W'_n$. Write $w_1 = us_is_{i-1}s_{i+1}s_{i+2} \cdots s_p, w_2 = us_is_{i-1}s_{i+1}s_{i+2} \cdots s_q, i \in [2, n-1], i + 2 \leq p \leq q \leq n - 1$, then $N' - w_1 \leq N' - w_2$.

Proof. We have $N' - us_is_{i-1} = N' - us_is_{i-1} s_{i+1} = u_{i-1} (i \geq 2)$.

$N' - us_is_{i-1} s_i = N' - us_is_{i-1} s_i s_{i+2} = \cdots = N' - us_is_{i-1} s_{i+1}s_{i+2} \cdots s_{n-1} = u_i (i \geq 2)$.

If we let $v_1 = s_is_{i-1}s_{i+1}s_{i+2} \cdots s_p, v_2 = s_is_{i-1}s_{i+1}s_{i+2} \cdots s_q, i + 2 \leq p \leq q \leq n - 1$, $w_1 = vw_1, w_2 = vw_2 \in W'_n$. We have $N' - w_1 = u_i = N' - w_2$. So, we have $N' - w_1 \leq N' - w_2$. □

Theorem 4.4. Let $w_1, w_2 \in W'_n$. Write $w_1 = us_is_{i-1}s_{i+1}s_{i+2} \cdots s_{i-1}s_{i+1}s_{i+2} \cdots s_2s_1, w_2 = us_is_{i-1}s_{i+1}s_{i+2} \cdots s_q = s_{i-1}s_{i+1}s_{i+2} \cdots s_2s_1, 2 \leq p \leq q$, then $N' - w_1 \leq N' - w_2$.

Proof. We have $N' - us_is_{i-1}s_{i+1}s_{i+2} \cdots s_{k-1}s_{k-1} = s_{k-1}u_{k-1}, k \in [2, n-1]$

If we let $v_1 = s_is_{i-1}s_{i+1}s_{i+2} \cdots s_{p-1}s_{p-2} \cdots s_2s_1, v_2 = s_is_{i-1}s_{i+1}s_{i+2} \cdots s_{q-1}s_{q-2} \cdots s_2s_1, 2 \leq p \leq q$, $w_1 = vw_1, w_2 = vw_2 \in W'_n$. We have $N' - w_1 = u_{p-1}u_p, N' - w_2 = u_{q-1}u_q$. By Lemma 3.2, we have $u_{p-1}u_p \leq u_{q-1}u_q$ since $p \leq q$. As a result, we have $N' - w_1 \leq N' - w_2$. □
\[ \text{Theorem 4.5.} \text{ Let } w_1, w_2 \in W'_n, w_1 \leq w_2. \text{ Write } w_2 = us_1s_2s_3 \cdots s_is_{i+1}s_is_{i+2} \cdots s_{i+r+k} \text{ with } i + 2 \leq i_1 < i_2 < \cdots < i_r < i_{r+1} < \cdots < i_{r+k} \leq n - 1, \ i \geq 2, \text{ where the hat denotes omission. Then } N' - w_1 \leq N' - w_2. \]

**Proof.** We write \[ w = us_1s_2s_3 \cdots s_is_{i+1}s_{i+2} \cdots s_{i+r} \text{ with } i_1 < i_2 < \cdots < i_r, \ i_1 \geq i + 2, \ i \geq 2. \]

We have \[ N' - w = u_i \text{ since } N' - us_1s_2s_3 \cdots s_i = u_i, \ N' - u_is_j = u_i \text{ with } j \geq i + 2. \]

So, if \( w_2 \in W'_n, \) we write:

\[ w_2 = us_1s_2s_3 \cdots s_is_{i+1}s_is_{i+2} \cdots s_{i+r+k} \]
\[ = u_is_1s_2s_3 \cdots s_is_{i+1}s_is_{i+2} \cdots s_{i+r+k} \]
\[ = s_2s_3 \cdots s_iu_is_{i+1}s_is_{i+2} \cdots s_{i+r+k} \]
\[ = s_2s_3 \cdots s_iu_is_{i+1}s_is_{i+2} \cdots s_{i+r+k} \]
\[ = s_2s_3 \cdots s_iu_is_{i+1}s_is_{i+2} \cdots s_{i+r+k} u_iu_1 \]

So, we have \( N' - w_2 = u_1u_i \) \( (i \geq 2) \)

Let \( w_1 \in W'_n, \) and \( w_1 \leq w_2, \) then:

(a1):

\[ w_1 = us_1 \cdots s_{j_1-1}s_{j_1} \cdots s_{j_2} \cdots s_{j_t} \cdots s_is_{i+1}s_is_{i+2} \cdots s_{i+r+k} \]

where \( j_1 < j_2 < \cdots < j_t, j_1, j_2, \cdots, j_t \in [1,i]. \) Then, we have \( N' - w_1 = u_1u_{j_1-1} \)

\( N' - w_2 = u_1u_i \)

(a2):

\[ w_1 = us_1 \cdots s_{j_1-1}s_{j_1} \cdots s_{j_2} \cdots s_{j_t} \cdots s_is_{i+1}s_is_{i+2} \cdots s_1s_2 \cdots s_{i+r+k} \]

where \( 1 \leq i_{q_1} < i_{q_2} < \cdots < i_{q_l} \leq i_{r+k}. \) So we have:

(i) If \( i_{q_1} = 1, \ N' - w_1 = u_1u_{j_1-1}u_1 = u_{j_1-1} < N' - w_2 = u_1u_i \)

(ii) If \( i_{q_1} > 1, \ N' - w_1 = u_1u_{j_1-1} < N' - w_2 = u_1u_i. \)

(b1):

\[ w_1 = us_1s_2 \cdots s_is_{i+1}s_is_{i+2} \cdots s_{i+p_1} \cdots s_{i+p_2} \cdots s_{i+k} \]

where \( i_1 \leq i_{p_1} < i_{p_2} < \cdots < i_{p_m} \leq i_r. \) Then, we have \( N' - w_1 = u_1u_i = N' - w_2. \)

(b2):

\[ w_1 = us_1s_2 \cdots s_is_{i+1}s_is_{i+2} \cdots s_{i+p_1} \cdots s_{i+p_2} \cdots s_{i+k} \]

where \( 1 \leq i_{q_1} < i_{q_2} < \cdots < i_{q_l} \leq i_{r+k}. \) So we have:

(i) If \( i_{q_1} = 1, \ N' - w_1 = u_1u_{u_1}u_1 = u_i < N' - w_2. \)

(ii) If \( i_{q_1} > 1, \ N' - w_1 = u_1u_i = N' - w_2. \)
The Bruhat ordering in Weyl group of type $D_n$  

\[(c1)\]:

\[w_1 = u s_1 \cdots s_{j_1-1} s_{j_1} \cdots s_{j_2} \cdots s_{j_t} s_{i_{t+1}} \cdots s_{i_p} \cdots s_{i_m} u s_1 s_{i_r+1} s_{i_r+2} \cdots s_{i_r+k}\]

where $j_1 < j_2 < \cdots < j_t, j_1, j_2, \ldots, j_t \in [1,t]$, $i_1 \leq i_p < i_{p_2} < \cdots < i_{p_m} \leq i_r$.

Then, we have $N' - w_1 = u_1 u_{j_1-1} < N' - w_2 = u_1 u_i$

\[(c2)\]:

\[w_1 = u s_1 \cdots s_{j_1-1} s_{j_1} \cdots s_{j_2} \cdots s_{j_t} s_{i_{t+1}} \cdots s_{i_p} \cdots s_{i_m} u s_1 \cdots s_{i_{q_1}} \cdots s_{i_{q_2}} \cdots s_{i_r} \cdots s_{i_r+k}\]

where $j_1 < j_2 < \cdots < j_t, j_1, j_2, \ldots, j_t \in [1,t]$, $i_1 \leq i_p < i_{p_2} < \cdots < i_{p_m} \leq i_r$,

this is same as $(c1)$, and $1 \leq i_{q_1} < i_{q_2} < \cdots < i_{q_r} \leq i_{r+k}$. So we have:

(i) If $i_{q_1} = 1$, $N' - w_1 = u_1 u_{j_1-1} u_1 = u_{j_1-1} < N' - w_2 = u_1 u_i$

(ii) If $i_{q_1} > 1$, $N' - w_1 = u_1 u_{j_1-1} < N' - w_2 = u_1 u_i$

(d):

\[w_1 = u s_1 s_2 s_3 \cdots s_{i_{q_1}} \cdots s_{i_{q_2}} \cdots s_{i_{q_t}} \cdots s_{i_{q_{r+k}}}{s_{i_{r+k}}}\]

where $1 \leq i_{q_1} < i_{q_2} < \cdots < i_{q_t} \leq i_{r+k}$. So we have:

(i) If $i_{q_1} = 1$, $N' - w_1 = u_1 u_{j_1-1} u_1 = u_i < N' - w_2 = u_1 u_i$

(ii) If $i_{q_1} > 1$, $N' - w_1 = u_1 u_i = N' - w_2$.

Above all, in this case, if $w_1 \leq w_2$, then $N' - w_1 \leq N' - w_2$. \(\square\)

**Theorem 4.6.** Let $w_1, w_2 \in W_n$, $w_1 \leq w_2$. Write $w_2 = s_{i_1} s_{i_2} \cdots s_{i_r} u s_{j_1} s_{j_2} \cdots s_{j_k}$, $\mathcal{L}(w_2) \neq \{u\}$, $s_{i_1}, s_{i_2}, \ldots, s_{i_r}, s_{j_1}, s_{j_2}, \ldots, s_{j_k} \in \{s_1, \ldots, s_{n-1}\}$ with $s_{j_1}, s_{j_2}, \ldots, s_{j_k}$ are distinct. Then $N' - w_1 \leq N' - w_2$.

**Proof.** We have $w_2 = s_{i_1} s_{i_2} \cdots s_{i_{p_1}} s_{j_{p_1}} s_{j_{p_2}} \cdots s_{j_{p_{t}}} u s_1 s_2 s_3 \cdots s_q$ with $t + q = k$ since the relations between $u, s_1, s_2, \ldots, s_{n-1}$.

So, $N' - w_2 = u_q$

Let $w_1 \in W_n'$, and $w_1 \leq w_2$, then:

(1): \[w_1 = s_1 \cdots s_{t} \cdot s_{j_{t+1}} \cdots s_{j_t} \cdot \hat{s}_{a} \cdot \hat{s}_{b} \cdot \hat{s}_{a_{j_{p_{t}}}} \cdot u s_2 s_3 \cdots s_q\]

where $s_a, s_b \in \{s_{i_1}, s_{i_2}, \ldots, s_{i_r}, s_{j_{p_{t}}}, \ldots, s_{j_{p_{t}}}\}$.

Then, we have $N' - w_1 = u_q = N' - w_2$

(2): \[w_1 = s_{i_1} s_{i_2} \cdots s_{i_r} s_{j_{p_1}} s_{j_{p_2}} \cdots s_{j_{p_{t}}} \cdot u s_2 \cdots \hat{s}_a \cdot \hat{s}_b \cdots \hat{s}_q\]

where $s_a, s_b \in \{s_2, s_3, \ldots, s_q\}$.

Then, we have $N' - w_1 = u_{a-1} < N' - w_2$.

(3): \[w_1 = s_{i_1} \cdots s_{i_2} \cdots s_{i_{q_1}} \cdots s_{i_{q_2}} \cdots s_{i_{q_{t}}} \cdot u s_2 \cdots \hat{s}_{a_2} \cdots \hat{s}_b \cdots \hat{s}_q\]

where $s_{a_2}, s_{b_2} \in \{s_1, s_{i_2}, \ldots, s_{i_r}, s_{j_{p_{t}}}, \ldots, s_{j_{p_{t}}}\}$, $s_{a_2}, s_{b_2} \in \{s_2, s_3, \ldots, s_q\}$.

Then, we have $N' - w_1 = u_{a_2-1} < N' - w_2$.

Above all, in this case, if $w_1 \leq w_2$, then $N' - w_1 \leq N' - w_2$. \(\square\)
References


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