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On the Strong Convergence of Split Equality Problems in Banach Spaces

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Abstract

In this paper, an iterative algorithm is introduced to solve split equality problems and the strong convergence of the sequence generated by the proposed iterative scheme is obtained in the framework of p-uniformly convex and uniformly smooth Banach spaces. The results presented in this paper are new.

Keywords: Split equality problems; Strong convergence; Banach spaces

1 Introduction

In this paper, we are concerned with the split equality problem (SEP) [1] which is formulated as finding points x and y with the property:

$$x \in C \quad and \quad y \in Q, \quad such \quad that \quad Ax = By, \tag{1.1}$$

where C and Q are two nonempty closed convex subsets of E_1 and E_2, respectively, $A : E_1 \to E_3, B : E_2 \to E_3$ are two bounded linear operators and E_1, E_2, E_3 are all p-uniformly convex and uniformly smooth Banach spaces. Let $\Omega = \{(x, y) : x \in C, y \in Q \quad such \quad that \quad Ax = By\}$ be the set of all solutions of (1.1). It's well known the SEP becomes a SFP, when $B = I$ and $E_2 = E_3$. The SFP has received much attention due to its applications in image reconstruction, signal processing, and intensity-modulated radiation therapy, see for instance [2-5].

For solving the SFP in p-uniformly convex Banach spaces which are also uniformly smooth, Schöpfer et al [9] proposed the following algorithm : giving $x_1 \in E_1$ and $n \geqslant 1$, set

$$x_{n+1} = \Pi_C J_{E_1}^* \left[J_{E_1}(x_n) - t_n A^* J_{E_2}(Ax_n - P_Q(Ax_n)) \right], \tag{1.2}$$

where Π_C denotes the Bregman projection and J the duality mapping. Clearly the algorithm(1.2) is the generalize of the Byrne's algorithm [10] in Hilbert space.

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), n \geqslant 1.$$

Recently, Wang [11] modified the algorithm (1.2) and proved strong convergence for the multiple-sets split feasibility problem ($MSSFP$). Very recently, Yekini Shehu [12] constructed another iterative scheme for solving the split feasibility problem (SFP), which does not involve the computation of Bregman projection onto the intersection of two half spaces at each step of the iteration, and obtained the strong convergence in p-uniformly convex and uniformly smooth Banach spaces.

In 2015, Wang [23] introduced a new hybrid Bregman projection iterative algorithm for Bregman quasi-strictly pseudo-contractive mapping and proved strong convergence result in reflexive Banach spaces. In 2017, Chen, Hu and Zeng [24] proposed a new hybrid projection method for solving split feasibility and fixed point problems involved in Bregman quasi-strictly pseudo-contractive mapping in p-uniformly convex and uniformly smooth real Banach spaces. Furthermore, they also obtained some strong convergence theorems.

For solving SEP (1.1), Byrne and Moudafi [6] put forward the alternating CQ-algorithm ($ACQA$) and the relaxed alternating CQ-algorithm ($RACQA$) and obtained weak convergence in Hilbert spaces. In order to approximate solutions of SEP, some investigators have been proposed the simultaneous iterative algorithm ($SSEA$), the relaxed $SSEA$ ($RSSEA$) and the perturbed $SSEA$ ($PSSEA$), for more details to see [7,8]. As we known, there are no one to study SEP in Banach space.

Our aim in this paper is to constructed an iterative scheme to solve SEP(1.1) and prove the strong convergence of the sequence proposed in p-uniformly convex and uniformly smooth Banach spaces. The results presented in this paper are new.

2 Preliminaries

Let E_1 and E_2 be real Banach spaces and let $A : E_1 \to E_2$ be a bounded linear operator. The *adjoint* operator of A, denoted by A^*, is a bounded linear operator defined by $A^*: E_2^* \to E_1^*$,

$$\langle A^*\bar{y}, x\rangle := \langle \bar{y}, Ax\rangle, \forall x \in E_1, \bar{y} \in E_2^*,$$

and the equalities $\|A^*\| = \|A\|$ and $\aleph(A^*) = \Re(A)^{\perp}$ are valid, where $\Re(A)^{\perp} := \{x^* \in E_2^* : \langle x^*, u\rangle = 0, \forall u \in \Re(A)\}$. For more details on bounded linear operators and their duals, please see [12-14].

Let $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let E be a real Banach space. The modulus of convexity $\delta_E : [0, 2] \to [0, 1]$ is defined as

$$\delta_E(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\}.$$

E is called uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$; p-uniformly convex if there is a $c_p > 0$ so that $\delta_E(\epsilon) \geq c_p\epsilon^p$ for any $\epsilon \in (0, 2]$. The modulus of smoothness $\rho_E(\tau) : [0, \infty) \to [0, \infty)$ is defined by

$$\rho_E(\tau) = \{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = \|y\| = 1\}.$$

E is called uniformly smooth if $\lim_{n\to\infty} \frac{\rho_E(\tau)}{\tau} = 0$; E is called q-uniformly smooth if there is a $C_q > 0$ so that $\rho_E(\tau) \leq C_q\tau^q$ for any $\tau > 0$. We assume that E is p-uniformly convex and uniformly smooth, which implies that its dual space E^* is q-uniformly smooth and uniformly convex. In this situation, it is known that the duality mapping J_E^p is one-to-one, single-valued and satisfies $J_E^p = (J_{E^*}^q)^{-1}$, where $J_{E^*}^q$ is the duality mapping of E^*.

Definition 2.1 ([12]). The duality mapping $J_p : E \to 2^{E^*}$ is defined by

$$J_{p(x)} = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}.$$

The duality mapping J_E^p is said to be weak-to-weak continuous if

$$x_n \rightharpoonup x \Rightarrow \langle J_E^p x_n, y\rangle \to \langle J_E^p x, y\rangle, \forall y \in E.$$

Lemma 2.2 ([15]). Let $x, y \in E$. If E is q-uniformly smooth, then there is a $C_q > 0$ so that

$$\|x - y\|^q \leq \|x\|^q - q\langle y, J_E^q(x)\rangle + C_q\|y\|^q.$$

Definition 2.3. Given a Gâteaux differentiable convex function $f : E \to R$. The Bregman distance with respect to f is defined as:

$$\Delta_f(x, y) := f(y) - f(x) - \langle f'(x), y - x\rangle, x, y \in E.$$

It is well know that the duality mapping J_p is the derivative of the function $f_p(x) = \frac{1}{p}\|x\|^p$. For the sake of convenience, the $\Delta_{f_p}(x, y)$ denoted by $\Delta_p(x, y)$, then the Bregman distance with respect to f_p now becomes

$$\Delta_{f_p}(x, y) = \frac{1}{q}(\|x\|^p - \|y\|^p) - \langle J_E^p x - J_E^p y, y\rangle.$$

The Bregman distance is not symmetric, therefore it is not a metric, but it posses the following important properties

$$\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle z - y, J_E^p x - J_E^p z\rangle, \tag{2.1}$$

and

$$\Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_E^p x - J_E^p y\rangle. \tag{2.2}$$

Likewise the definition of metric projection, one can define the Bregman projection:

$$\Pi_C x = argmin_{y \in C} \Delta_p(x, y), x \in E,$$

as the unique minimizer of the Bregman distance (see[18]).The Bregman projection can also be characterized by the following variational inequality:

$$\langle J_E^p x - J_E^p(\Pi_C x), z - \Pi_C x\rangle \leq 0, \forall z \in C. \tag{2.3}$$

It follows from (2.3) that

$$\Delta_p(\Pi_C x, z) \leq \Delta_p(x, z) - \Delta_p(x, \Pi_C x), \forall z \in C. \tag{2.4}$$

Lemma 2.4 ([9]). Let $x, y \in E$. If E is p-uniformly convex space, the metric and Bregman distance have the following relations:

$$\tau\|x - y\|^p \leq \Delta_p(x, y) \leq \langle x - y, J_E^p x - J_E^p y\rangle,$$

where $\tau > 0$ is some fixed number.

Remark 2.5. Let E be a p-uniformly convex and uniformly smooth Banach space and let $\{x_n\}$,$\{y_n\}$ be two sequences of E. From Lemma 2.4, we know that $\Delta_p(x_n, y_n) \to 0$ if and only if $\|x_n - y_n\| \to 0$.

Following [20,21], we make use of the function $V_p : E^* \times E \to [0, +\infty)$ associated with f_p, which is defined by

$$V_p(\bar{x}, x) = \frac{1}{q}\|\bar{x}\|^q - \langle \bar{x}, x\rangle + \frac{1}{p}\|x\|^p, \forall x \in E, \bar{x} \in E^*.$$

Then V_p is nonnegative and

$$V_p(\bar{x}, x) = \Delta_p(J_{E^*}^p(\bar{x}), x) = \Delta_p(J_E^q(\bar{x}), x), \tag{2.5}$$

for all $x \in E$ and $\bar{x} \in E^*$. Moreover, by the subdifferential inequality,

$$V_p(\bar{x}, x) + \langle \bar{y}, J_E^*(\bar{x}) - x \rangle \leqslant V_p(\bar{x} + \bar{y}, x). \tag{2.6}$$

for all $x \in E$ and $\bar{x}, \bar{y} \in E^*$(see [16], [22]). In addition, V_p is convex in the first variable. Thus, for all $z \in E$

$$\Delta_p(J_{E^*}^q(\sum_{i=1}^N t_i J_E^p x_i), z) \leqslant \sum_{i=1}^N t_i \Delta_p(x_i, z). \tag{2.7}$$

Lemma 2.6 ([15]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 1.$$

if (i) $\{\alpha_n\} \subset [0,1], \sum \alpha_n = \infty$; (ii) $\limsup \sigma_n < 0$; (iii) $\gamma_n \geq 0 (n \geq 1), \sum \gamma_n < \infty$, then $a_n \to 0$ as $n \to \infty$.

Lemma 2.7 ([19]). Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in N$. Then there exists a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in N$:

$$a_{m_k} \leq a_{m_k+1} \quad and \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$

3 Main Results

Theorem 3.1. Let E_1, E_2 and E_3 be three p-uniformly convex and uniformly smooth Banach spaces. Let C and Q be nonempty closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_3$, $B : E_2 \to E_3$ be two bounded linear operators and $A^* : E_3^* \to E_1^*$, $B^* : E_3^* \to E_2^*$ be the adjoint operators of A and B, respectively. For a fixed $u \in C$ and a fixed $v \in Q$, the sequence $\{(x_n, y_n)\}$ is generated by

$$\begin{cases} u_n = J_{E_1^*}^q[J_{E_1}^p x_n - \lambda_n A^* J_{E_3}^p (Ax_n - By_n)] \\ v_n = J_{E_2^*}^q[J_{E_2}^p y_n + \lambda_n B^* J_{E_3}^p (Ax_n - By_n)] \\ x_{n+1} = \Pi_C J_{E_1^*}^q(\beta_n J_{E_1}^p u + (1 - \beta_n) J_{E_1}^p u_n) \\ y_{n+1} = \Pi_Q J_{E_2^*}^q(\beta_n J_{E_2}^p v + (1 - \beta_n) J_{E_2}^p v_n), \end{cases} \tag{3.1}$$

where the following conditions are satisfied:

(i) $\{\beta_n\} \subset (0,1)$ and $\lim_{n\to\infty} \beta_n = 0$;

(ii) $\sum_{n=1}^{\infty}\beta_n=\infty$;

(iii) $0<2L\leqslant\lambda_n\leqslant M$, where $M=\max\{(\frac{q}{C_q\|A\|^q})^{\frac{1}{q-1}},(\frac{q}{C_q\|B\|^q})^{\frac{1}{q-1}}\}$, and $L=\max\{\frac{C_q(\lambda_n\|A\|)^q}{q},\frac{C_q(\lambda_n\|B\|)^q}{q}\}$.

If the Ω is nonempty, then the sequence $\{(x_n,y_n)\}$ converges strongly to a solution of the $SEP(1.1)$.

Proof. Let $(x,y)\in\Omega$ and $e_n:=Ax_n-By_n,\forall n\geqslant 1$. From (3.1) and (2.7), we have

$$\begin{aligned}\Delta_p(x_{n+1},x)&\leqslant\Delta_p(J^q_{E_1^*}(\beta_nJ^p_{E_1}u+(1-\beta_n)J^p_{E_1}u_n),x)\\&\leqslant\beta_n\Delta_p(u,x)+(1-\beta_n)\Delta_p(u_n,x).\end{aligned}\tag{3.2}$$

It follows from Lemma 2.1 and the definition of Δ_p that

$$\begin{aligned}\Delta_p(u_n,x)&=\Delta_p(J^q_{E_1^*}[J^p_{E_1}x_n-\lambda_nA^*J^p_{E_3}(Ax_n-By_n)],x)\\&=\tfrac{1}{q}\|J^p_{E_1}x_n-\lambda_nA^*J^p_{E_3}(Ax_n-By_n)\|^q-\langle J^p_{E_1}x_n,x\rangle\\&\quad+\lambda_n\langle J^p_{E_3}(Ax_n-By_n),Ax\rangle+\tfrac{1}{p}\|x\|^p\\&\leqslant\tfrac{1}{q}\|J^p_{E_1}x_n\|^q-\lambda_n\langle Ax_n,J^p_{E_3}e_n\rangle+\tfrac{C_q(\lambda_n\|A\|)^q}{q}\|J^p_{E_3}e_n\|^q\\&\quad-\langle J^p_{E_1}x_n,x\rangle+\lambda_n\langle J^p_{E_3}e_n,Ax\rangle+\tfrac{1}{p}\|x\|^p\\&=\tfrac{1}{q}\|x_n\|^p-\langle J^P_{E_1}x_n,x\rangle+\tfrac{1}{p}\|x\|^p+\lambda_n\langle J^p_{E_3}e_n,Ax-Ax_n\rangle+\tfrac{C_q(\lambda_n\|A\|)^q}{q}\|e_n\|^p\\&=\Delta_p(x_n,x)+\lambda_n\langle J^p_{E_3}e_n,Ax-Ax_n\rangle+\tfrac{C_q(\lambda_n\|A\|)^q}{q}\|e_n\|^p.\end{aligned}\tag{3.3}$$

So, it follows from (3.2) and (3.3) that

$$\begin{aligned}\Delta_p(x_{n+1},x)&\leqslant\beta_n\Delta_p(u,x)+(1-\beta_n)\Delta_p(x_n,x)\\&\quad+(1-\beta_n)\lambda_n\langle J^p_{E_3}e_n,Ax-Ax_n\rangle+(1-\beta_n)\frac{C_q(\lambda_n\|A\|)^q}{q}\|e_n\|^p\\&\leqslant\beta_n\Delta_p(u,x)+(1-\beta_n)\Delta_p(x_n,x)\\&\quad+(1-\beta_n)\lambda_n\langle J^p_{E_3}e_n,Ax-Ax_n\rangle+(1-\beta_n)L\|e_n\|^p.\end{aligned}\tag{3.4}$$

Similarly, we have

$$\begin{aligned}\Delta_p(y_{n+1},y)&\leqslant\beta_n\Delta_p(v,y)+(1-\beta_n)\Delta_p(y_n,y)\\&\quad+(1-\beta_n)\lambda_n\langle J^p_{E_3}e_n,By_n-By\rangle+(1-\beta_n)L\|e_n\|^p.\end{aligned}\tag{3.5}$$

Since $Ax=By$, adding up (3.4) and (3.5), we can get

$$\begin{aligned}&\Delta_p(x_{n+1},x)+\Delta_p(y_{n+1},y)\\&\quad\leqslant\beta_n(\Delta_p(u,x)+\Delta_p(v,y))+(1-\beta_n)(\Delta_p(x_n,x)+\Delta_p(y_n,y))\\&\qquad-(1-\beta_n)\lambda_n\langle J^p_{E_3}e_n,e_n\rangle+2(1-\beta_n)L\|e_n\|^p\\&\quad\leqslant\beta_n(\Delta_p(u,x)+\Delta_p(v,y))+(1-\beta_n)(\Delta_p(x_n,x)+\Delta_p(y_n,y))\end{aligned}$$

$$-(1-\beta_n)(\lambda_n-2L)\|e_n\|^p. \tag{3.6}$$

Now, setting $W_n(x,y)=\Delta_p(x_n,x)+\Delta_p(y_n,y)$, then from conditions $(i),(iii)$ and (3.6) we have

$$\begin{aligned}
W_{n+1}(x,y) &\leqslant (1-\beta_n)W_n(x,y)+\beta_n(\Delta_p(u,x)+\Delta_p(v,y))-(1-\beta_n)(\lambda_n-2L)\|e_n\|^p\\
&\leqslant (1-\beta_n)W_n(x,y)+\beta_n(\Delta_p(u,x)+\Delta_p(v,y))\\
&\leqslant \max\{W_n(x,y),(\Delta_p(u,x)+\Delta_p(v,y))\}\\
&\ \vdots\\
&\leqslant \max\{W_1(x,y),(\Delta_p(u,x)+\Delta_p(v,y))\}.
\end{aligned} \tag{3.7}$$

Hence, $\{W_n(x,y)\}$ is bounded for any $(x,y)\in\Omega$.

The rest of the proof will be divided into two cases.

Case 1. Let $(x^*,y^*)\in\Omega$. Suppose that there exists $n_0\in N$ such that $\{W_n(x^*,y^*)\}$ is monotonically non-increasing as $n\geq n_0$, then $\{W_n(x^*,y^*)\}$ converges. From (3.6), we can get that

$$\begin{aligned}
&(1-\beta_n)(\lambda_n-2L)\|e_n\|^p\\
&\quad\leqslant (1-\beta_n)[(\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)]-[(\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*)]\\
&\qquad+\beta_n(\Delta_p(u,x^*)+\Delta_p(v,y^*))\\
&\quad\leqslant W_n(x^*,y^*)-W_{n+1}(x^*,y^*)+\beta_n(\Delta_p(u,x^*)+\Delta_p(v,y^*)).
\end{aligned} \tag{3.8}$$

By condition $(i),(iii)$ we have

$$\begin{aligned}
&0<(1-\beta_n)(\lambda_n-2L)\|e_n\|^p\\
&\quad\leqslant W_n(x^*,y^*)-W_{n+1}(x^*,y^*)+\beta_n(\Delta_p(u,x^*)+\Delta_p(v,y^*))\to 0, n\to\infty,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|Ax_n-By_n\|=0. \tag{3.9}$$

Since $u_n=J^q_{E_1^*}[J^p_{E_1}x_n-\lambda_nA^*J^p_{E_3}(Ax_n-By_n)]$, then we have

$$\begin{aligned}
0\leqslant\|J^p_{E_1}u_n-J^p_{E_1}x_n\| &\leqslant \lambda_n\|A^*\|\|J^p_{E_3}(Ax_n-By_n)\|\\
&\leqslant \left(\frac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}\|A^*\|\|Ax_n-By_n\|\\
&\leqslant M\|A^*\|\|Ax_n-By_n\|\to 0, n\to\infty.
\end{aligned}$$

Hence, we obtain

$$\lim_{n\to\infty}\|J^p_{E_1}u_n-J^p_{E_1}x_n\|=0. \tag{3.10}$$

Since $J^q_{E_1^*}$ is also norm-to-norm uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.11}$$

Similarly, we can get

$$\lim_{n\to\infty}\|J_{E_2}^p v_n - J_{E_2}^p y_n\| = 0. \tag{3.12}$$

Since $J_{E_2^*}^q$ is also norm-to-norm uniformly continuous on bounded subsets of E_2^*, we have

$$\lim_{n\to\infty}\|v_n - y_n\| = 0. \tag{3.13}$$

Since $\Delta_p(x_n, x^*) \leqslant W_n(x^*, y^*), \Delta_p(y_n, y^*) \leqslant W_n(x^*, y^*)$ and $\lim_{n\to\infty} W_n(x^*, y^*)$ exists, we know that $\{(x_n.y_n)\}$ is bounded. So there exists a subsequence $\{(x_{n_j}, y_{n_i})\}$ of $\{(x_n, y_n)\}$ such that $\{(x_{n_j}, y_{n_i})\} \rightharpoonup (\bar{x}, \bar{y})$ since E_1 and E_2 are p-uniformly convex and reflexive. Next, we will prove $\bar{x} \in C$ and $\bar{y} \in Q$. From Definition 2.2, Lemma 2.4 and (2.3) we have

$$\begin{aligned}
\Delta_p(\bar{x}, \Pi_C\bar{x}) &\leqslant \langle J_{E_1}^p\bar{x} - J_{E_1}^p\Pi_C\bar{x}, \bar{x} - \Pi_C\bar{x}\rangle \\
&= \langle J_{E_1}^p\bar{x} - J_{E_1}^p\Pi_C\bar{x}, \bar{x} - x_{n_j}\rangle + \langle J_{E_1}^p\bar{x} - J_{E_1}^p\Pi_C\bar{x}, x_{n_j} - u_{n_j}\rangle \\
&\quad + \langle J_{E_1}^p\bar{x} - J_{E_1}^p\Pi_C\bar{x}, u_{n_j} - \Pi_C\bar{x}\rangle \\
&\leqslant \langle J_{E_1}^p\bar{x} - J_{E_1}^p\Pi_C\bar{x}, \bar{x} - x_{n_j}\rangle + \langle J_{E_1}^p\bar{x} - J_{E_1}^p\Pi_C\bar{x}, x_{n_j} - u_{n_j}\rangle.
\end{aligned}$$

As $j \to \infty$, we have $\Delta_p(\bar{x}, \Pi_C\bar{x})$=0, i.e. $\bar{x} \in C$. Similarly, we have $\bar{y} \in Q$.

Since A and B are bounded linear operators, we know that $A\bar{x} - B\bar{y}$ is a weak cluster point of $\{Ax_n - By_n\}$. From the weakly lower semi-continuous property of the norm and (3.9), we get

$$\|A\bar{x} - B\bar{y}\| \leqslant \liminf_{n\to\infty}\|Ax_n - By_n\| = 0.$$

So, $A\bar{x} = B\bar{y}$, this implies $(\bar{x}, \bar{y}) \in \Omega$.

Finally, we prove the strong convergence $\{(x_n, y_n)\}$. Then from (2.5) and (2.6) we can obtain

$$\begin{aligned}
\Delta_p(x_{n+1}, \bar{x}) &\leqslant \Delta_p(J_{E_1^*}^q(\beta_n J_{E_1}^p u + (1-\beta_n)J_{E_1}^p u_n), \bar{x}) \\
&= V_p(\beta_n J_{E_1}^p u + (1-\beta_n)J_{E_1}^p u_n, \bar{x}) \\
&\leqslant V_p(\beta_n J_{E_1}^p u + (1-\beta_n)J_{E_1}^p u_n - \beta_n(J_{E_1}^p u - J_{E_1}^p\bar{x}), \bar{x}) \\
&\quad + \langle J_{E_1^*}^q(\beta_n J_{E_1}^p u + (1-\beta_n)J_{E_1}^p u_n - \bar{x}, -\beta_n(J_{E_1}^p u - J_{E_1}^p\bar{x})\rangle \\
&= V_p(\beta_n J_{E_1}^p\bar{x} + (1-\beta_n)J_{E_1}^p u_n, \bar{x}) + \beta_n\langle\omega_n - \bar{x}, J_{E_1}^p u - J_{E_1}^p\bar{x}\rangle \\
&= \Delta_p(J_{E_1^*}^q(\beta_n J_{E_1}^p\bar{x} + (1-\beta_n)J_{E_1}^p u_n), \bar{x}) + \beta_n\langle\omega_n - \bar{x}, J_{E_1}^p u - J_{E_1}^p\bar{x}\rangle \\
&\leqslant \beta_n\Delta_p(\bar{x}, \bar{x}) + (1-\beta_n)\Delta_p(u_n, \bar{x}) + \beta_n\langle\omega_n - \bar{x}, J_{E_1}^p u - J_{E_1}^p\bar{x}\rangle \\
&= (1-\beta_n)\Delta_p(u_n, \bar{x}) + \beta_n\langle\omega_n - \bar{x}, J_{E_1}^p u - J_{E_1}^p\bar{x}\rangle,
\end{aligned} \tag{3.14}$$

where $\omega_n := J_{E_1^*}^q(\beta_n J_{E_1}^p u + (1-\beta_n)J_{E_1}^p u_n)$. From (3.1) we get

$$\begin{aligned}
\Delta_p(\omega_n, u_n) &= \Delta_p(J_{E_1^*}^q(\beta_n J_{E_1}^p u + (1-\beta_n)J_{E_1}^p u_n), u_n) \\
&\leqslant \beta_n\Delta_p(u, u_n) + (1-\beta_n)\Delta_p(u_n, u_n) \\
&= \beta_n\Delta_p(u, u_n) \to 0, n \to \infty.
\end{aligned} \tag{3.15}$$

It follows from the Remark 2.5 that $\lim_{n\to\infty}\|\omega_n - u_n\| = 0$, and from (3.11) we have

$$\|\omega_n - x_n\| \leqslant \|\omega_n - u_n\| + \|u_n - x_n\| \to 0, n \to \infty,$$

which implies that

$$\lim_{n\to\infty}\|\omega_n - x_n\| = 0. \tag{3.16}$$

Since $x_{n_j} \rightharpoonup \bar{x}$, so we have $\omega_{n_j} \rightharpoonup \bar{x}$. Furthermore,

$$\limsup_{n\to\infty}\langle \omega_n - \bar{x}, J^p_{E_1}u - J^p_{E_1}\bar{x}\rangle = \lim_{j\to\infty}\langle \omega_{n_j} - \bar{x}, J^p_{E_1}u - J^p_{E_1}\bar{x}\rangle \leqslant 0. \tag{3.17}$$

Submitting (3.17) into (3.14), we obtain

$$\begin{aligned}\Delta_p(x_{n+1},\bar{x}) &\leqslant (1-\beta_n)\Delta_p(u_n,\bar{x}) + \beta_n\langle \omega_n - \bar{x}, J^p_{E_1}u - J^p_{E_1}\bar{x}\rangle\\ &\leqslant (1-\beta_n)\Delta_p(x_n,\bar{x}) + (1-\beta_n)\lambda_n\langle J^p_{E_3}e_n, A\bar{x} - Ax_n\rangle\\ &\quad +(1-\beta_n)\frac{C_q(\lambda_n\|A\|)^q}{q}\|e_n\|^p + \beta_n\langle \omega_n - \bar{x}, J^p_{E_1}u - J^p_{E_1}\bar{x}\rangle. \quad (3.18)\end{aligned}$$

Similary, we have

$$\begin{aligned}\Delta_p(y_{n+1},\bar{y}) &\leqslant (1-\beta_n)\Delta_p(y_n,\bar{y}) + (1-\beta_n)\lambda_n\langle J^p_{E_3}e_n, By_n - B\bar{y}\rangle\\ &\quad +(1-\beta_n)\frac{C_q(\lambda_n\|B\|)^q}{q}\|e_n\|^p + \beta_n\langle \delta_n - \bar{y}, J^p_{E_2}v - J^p_{E_2}\bar{y}\rangle, \quad (3.19)\end{aligned}$$

where $\delta_n := J^q_{E_1^*}(\beta_n J^p_{E_2}v + (1-\beta_n)J^p_{E_2}v_n)$.

Adding up (3.18) and (3.19), we have

$$\begin{aligned}W_{n+1}(\bar{x},\bar{y}) &= \Delta_p(x_{n+1},\bar{x}) + \Delta_p(y_{n+1},\bar{y})\\ &\leqslant (1-\beta_n)W_n(\bar{x},\bar{y}) - (1-\beta_n)(\lambda_n - 2L)\|e_n\|^p\\ &\quad +\beta_n[\langle \omega_n - \bar{x}, J^p_{E_1}u - J^p_{E_1}\bar{x}\rangle + \langle \delta_n - \bar{y}, J^p_{E_2}v - J^p_{E_2}\bar{y}\rangle]\\ &\leqslant (1-\beta_n)W_n(\bar{x},\bar{y}) + \beta_n[\langle \omega_n - \bar{x}, J^p_{E_1}u - J^p_{E_1}\bar{x}\rangle\\ &\quad +\langle \delta_n - \bar{y}, J^p_{E_2}v - J^p_{E_2}\bar{y}\rangle]. \quad (3.20)\end{aligned}$$

It follows from Lemma 2.6 that $\lim_{n\to\infty}W_n(\bar{x},\bar{y}) = 0$, which implies that

$$\Delta_p(x_n,\bar{x}) + \Delta_p(y_n,\bar{y}) \to 0, n \to \infty.$$

It follows from Lemma 2.4 and Remark 2.5 that $x_n \to \bar{x}$ and $y_n \to \bar{y}$ as $n \to \infty$. That is the sequence $\{(x_n, y_n)\}$ converges strongly to a solution $(\bar{x},\bar{y}) \in \Omega$.

Case 2. Assume that $\{W_n(\bar{x},\bar{y})\}$ is not monotonically decreasing sequence. Let $\tau : N \to N$ be a mapping for all $n \geqslant n_0$ by

$$\tau(n) := \max\{k \in N : k \leqslant n, W_k \leqslant W_{k+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leqslant \ W_{\tau(n)} \leqslant W_{\tau(n)+1}, \forall n \geqslant n_0. \tag{3.21}$$

The same as in case 1, we have the following equalities or inequalities

$$\lim_{n\to\infty} \|Ax_{(n)} - By_{\tau(n)}\| = 0, \quad \lim_{n\to\infty} \|A^* J^p_{E_3}(Ax_{\tau(n)} - By_{\tau(n)})\| = 0,$$

$$\lim_{n\to\infty} \|B^* J^p_{E_3}(Ax_{\tau(n)} - By_{\tau(n)})\| = 0, \lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0, \lim_{n\to\infty} \|y_{\tau(n)+1} - y_{\tau(n)}\| = 0,$$

$$\limsup_{n\to\infty} \langle \omega_{\tau(n)} - \bar{x}, J^p_{E_1} u - J^p_{E_1} \bar{x}\rangle \leqslant 0, \quad \limsup_{n\to\infty} \langle \delta_{\tau(n)} - \bar{y}, J^p_{E_2} v - J^p_{E_2} \bar{y}\rangle \leqslant 0.$$

Since $\{(x_{\tau(n)}, y_{\tau(n)})\}$ is bounded, there exists s subsequence of $\{(x_{\tau(n)}, y_{\tau(n)})\}$ which converges weakly to (x^*, y^*), it is obvious that $x^* \in C, y^* \in Q$ and $(x^*, y^*) \in \Omega$. From (3.20), we know that

$$\begin{aligned} W_{\tau(n)+1}(x^*, y^*) \ &\leqslant \ (1 - \beta_{\tau(n)}) W_{\tau(n)}(x^*, y^*) \\ &\quad + \beta_{\tau(n)}[\langle \omega_{\tau(n)} - x^*, J^p_{E_1} u - J^p_{E_1} x^*\rangle + \langle \delta_{\tau(n)} - y^*, J^p_{E_2} v - J^p_{E_2} y^*\rangle]. \end{aligned}$$

From (3.21), we have

$$\begin{aligned} W_{\tau(n)}(x^*, y^*) \ &\leqslant \ W_{\tau(n)+1}(x^*, y^*) \\ &\leqslant \ (1 - \beta_{\tau(n)}) W_{\tau(n)}(x^*, y^*) + \beta_{\tau(n)}[\langle \omega_{\tau(n)} - x^*, J^p_{E_1} u - J^p_{E_1} x^*\rangle \\ &\quad + \langle \delta_{\tau(n)} - y^*, J^p_{E_2} v - J^p_{E_2} y^*\rangle], \end{aligned}$$

i.e.

$$W_{\tau(n)}(x^*, y^*) \leqslant \langle \omega_{\tau(n)} - x^*, J^p_{E_1} u - J^p_{E_1} x^*\rangle + \langle \delta_{\tau(n)} - y^*, J^p_{E_2} v - J^p_{E_2} y^*\rangle.$$

Then from (3.19) we have $\limsup_{n\to\infty} W_{\tau(n)}(x^*, y^*) \leqslant 0$. Further, we know that $\lim_{n\to\infty} \Delta_p(x_{\tau(n)}, x^*) = 0$, $\lim_{n\to\infty} \Delta_p(y_{\tau(n)}, y^*) = 0$. So

$$\lim_{n\to\infty} \|x_{\tau(n)} - x^*\| = 0, \lim_{n\to\infty} \|y_{\tau(n)} - y^*\| = 0.$$

It follows from $\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0$ and $\lim_{n\to\infty} \|y_{\tau(n)+1} - y_{\tau(n)}\| = 0$ that $\lim_{n\to\infty} \|x_{\tau(n)+1} - x^*\| = 0, \lim_{n\to\infty} \|y_{\tau(n)+1} - y^*\| = 0$. Now, by (2.5), we have that

$$\begin{aligned} 0 \leqslant W_{\tau(n)+1} \ &\leqslant \ \langle x_{\tau(n)+1} - x^*, J^p_{E_1} x_{\tau(n)+1} - J^p_{E_1} x^*\rangle + \langle y_{\tau(n)+1} - y^*, J^p_{E_1} y_{\tau(n)+1} - J^p_{E_1} y^*\rangle \\ &\leqslant \ \|x_{\tau(n)+1} - x^*\| \|J^p_{E_1} x_{\tau(n)+1} - J^p_{E_1} x^*\| \\ &\quad + \|y_{\tau(n)+1} - y^*\| \|J^p_{E_2} y_{\tau(n)+1} - J^p_{E_2} y^*\| \to 0, \quad n \to \infty. \end{aligned}$$

For $n \geqslant n_0$, it is easy to see that $W_{\tau(n)} \leqslant W_{\tau(n)+1}$ if $n \neq \tau(n)$(that is, $\tau(n) < n$), because $W_j \geqslant W_{j+1}$ for $\tau(n) + 1 \leqslant j \leqslant n$. As a consequence, for all $n \geqslant n_0$,

$$0 \leqslant W_n \leqslant \max\{W_{\tau(n)}, W_{\tau(n)+1}\} = W_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} W_n = 0$, that is $\{(x_n, y_n)\}$ converges strongly to (x^*, y^*). This completes the proof.

Corollary 3.2. Let E_1, E_2 and E_3 be three L_p spaces with $2 \leqslant p < \infty$. Let C and Q be nonempty closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_3$, $B : E_2 \to E_3$ be two bounded linear operators and $A^* : E_3^* \to E_1^*$, $B^* : E_3^* \to E_2^*$ be the adjoint of A and B, respectively. For a fixed $u \in C$ and a fixed $v \in Q$, the sequence $\{(x_n, y_n)\}$ is generated by

$$\begin{cases} u_n = J^q_{E_1^*}[J^p_{E_1} x_n - \lambda_n A^* J^p_{E_3}(Ax_n - By_n)] \\ v_n = J^q_{E_2^*}[J^p_{E_2} y_n + \lambda_n B^* J^p_{E_3}(Ax_n - By_n)] \\ x_{n+1} = \Pi_C J^q_{E_1^*}(\beta_n J^p_{E_1} u + (1-\beta_n) J^p_{E_1} u_n) \\ y_{n+1} = \Pi_Q J^q_{E_2^*}(\beta_n J^p_{E_2} v + (1-\beta_n) J^p_{E_2} v_n), \end{cases}$$

where the following conditions are satisfied:

(i) $\{\beta_n\} \subset (0,1)$ and $\lim_{n\to\infty} \beta_n = 0$;

(ii) $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $0 < 2L \leqslant \lambda_n \leqslant M$, where $M = \max\{(\frac{q}{C_q\|A\|^q})^{\frac{1}{q-1}}, (\frac{q}{C_q\|B\|^q})^{\frac{1}{q-1}}\}$, and $L = \max\{\frac{C_q(\lambda_n\|A\|)^q}{q}, \frac{C_q(\lambda_n\|B\|)^q}{q}\}$.

If the Ω is nonempty, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution of the SEP (1.1).

Corollary 3.3. Let H_1, H_2 and H_3 be three real Hilbert spaces. Let C and Q be nonempty closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators and A^*, B^* be the adjoint of A and B, respectively. For a fixed $u \in C$ and a fixed $v \in Q$, the sequence $\{(x_n, y_n)\}$ is generated by

$$\begin{cases} u_n = x_n - \lambda_n A^*(Ax_n - By_n)] \\ v_n = y_n + \lambda_n B^* J^p_{E_3}(Ax_n - By_n)] \\ x_{n+1} = P_C(\beta_n u + (1-\beta_n) u_n) \\ y_{n+1} = P_Q(\beta_n v + (1-\beta_n) v_n), \end{cases}$$

where the following conditions are satisfied:

(i) $\{\beta_n\} \subset (0,1)$ and $\lim_{n\to\infty} \beta_n = 0$;

(ii) $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $0 < \lambda_n < \frac{1}{\max\{\lambda_A^2, \lambda_B^2\}}$, where λ_A, λ_B stand for the spectral radius of A and B.

If Ω is nonempty, then the sequence $\{(x_n, y_n)\}$ converges strongly to a solution of the SEP (1.1).

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