Sequences from Heptagonal Pyramid
Corners of Integer

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Abstract
This paper presents a number of sequences from the heptagonal pyramid corners. The corner sequence is obtained by using a 3-dimensional sketch of the heptagon pyramid. There are 7 sequences of corner from the heptagonal pyramid. The number of sequences of corners on the pyramid is equal to the number of pyramidal base facets.

Keywords: Integer sequences, sequences of pyramids, corners heptagonal pyramids, heptagonal pyramids

1 Introduction
A sequence is a number arranged according to sequence and certain pattern. Protter [8] says that if the domain of a sequence $a$ is the set of all natural numbers (so that the sequence is infinite), we denote the sequence by $a_1, a_2, a_3, \ldots, a_n$. There are many references that discuss the integer sequences, among others are Aksoy and Khamsi [1], Protter [8] and Gulliver [3, 4, 5, 6, 7]. Gulliver [7] finds a number of sequences based on integers arranged in a fractal like structure. Gulliver [3] in his paper finds several sequences of numbers formed from the array of integers in the tetrahedron. Using the illustrations in Figure 1 Gulliver [3] gets some sequences based on certain criteria such as: the corners of the tetrahedron, the number of tetrahedron surfaces, the number of...
rows on the base of the tetrahedron at each level and other criteria. Based on Figure 1 the following sequence are obtained from the corners of the pyramid:

\[ u_n = \frac{1}{6} (n + 2)(n^2 - 2n + 3). \] (1)

\[ u_n = \frac{1}{6} (n^3 + 3n^2 - 4n + 6). \] (2)

\[ u_n = \frac{1}{6} (n^3 + 3n^2 - 4n + 6). \] (3)

These three sequences (1), (2) and (3) are registered in OEIS of Sloane [10] with the list number A050407, A105163 and A000292. Gulliver [4] also finds some new sequences of integers arranged on the pyramid using the sketch. There are four sequences from the corners on the pyramid, i.e.

\[ u_n = \frac{1}{6} (2n^3 - 3n^2 + n + 6). \]

\[ u_n = \frac{1}{6} (2n^3 - 3n^2 + 7n). \]

\[ u_n = \frac{1}{6} (2n^3 + 3n^2 - 5n + 6). \]

\[ u_n = \frac{1}{6} (2n^3 + 3n^2 + n). \]

For the above case the authors give another illustration presented in Figure 2 for tetrahedron and Figure 3 for the pyramid.

Then Gulliver [5] discusses the sequence obtained from the array of integers on the pentagonal pyramid. The corners sequence obtained on the pentagonal
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Figure 2: A tetrahedron array of integers

Figure 3: Sequences from the corner of pyramid
pyramid as follows:

\[ u_n = \frac{1}{2} \left( n^3 - 2n^2 + n + 2 \right) \]

\[ u_n = \frac{1}{2} \left( n^3 + n^2 - 6n + 6 \right) \]

\[ u_n = \frac{1}{2} \left( n^3 + n^2 - 4n + 4 \right) \]

\[ u_n = \frac{1}{2} \left( n^3 + n^2 - 2n + 2 \right) \]

\[ u_n = \frac{1}{2} \left( n^3 + n^2 \right) \]

Furthermore Gulliver [6] finds a new sequence of numbers in the pyramid round integer arrangement. The corner sequences obtained on the hexagonal pyramid are

\[ u_n = \frac{1}{6} \left( 4n^3 - 9n^2 + 5n + 6 \right) \] (4)

\[ u_n = \frac{1}{6} \left( 4n^3 + 3n^2 - 25n + 24 \right) \] (5)

\[ u_n = \frac{1}{6} \left( 4n^3 + 3n^2 - 19n + 18 \right) \] (6)

\[ u_n = \frac{1}{6} \left( 4n^3 + 3n^2 - 13n + 12 \right) \] (7)

\[ u_n = \frac{1}{6} \left( 4n^3 + 3n^2 - 7n + 6 \right) \] (8)

\[ u_n = \frac{1}{6} \left( 4n^3 + 3n^2 - n \right) \] (9)

The five sequences (4), (5), (6), (7) and (8) are not yet registered in the database [10]. While the last sequences have been registered in OEIS, the name is sequences is hexagonal pyramidal numbers or greengrocer’s numbers.

In this paper we obtain the corner sequences for the heptagonal pyramid by using a 3-dimensional sketch of a heptagonal pyramid.

## 2 The Main Results

A three-dimensional sketch of the heptagonal pyramid is presented in Figure 4 with a regularly arranged integer on the pyramid. The arrangement of integers can generate a number of sequences. The following gives a construction of the sequence based on the corner sequence in a heptagonal pyramid.

Based on the Figure 4 the following sequences are obtained from the corner of the pyramids, i.e.
Sequences from heptagonal pyramid corners of integer

Figure 4: Sequences from the corners heptagonal pyramids

(i) $1, 2, 9, 27, 61, \ldots$ or $u_n = \frac{5}{6}n^3 - 2n^2 + \frac{7}{6}n + 1$.

(ii) $1, 3, 16, 45, 95, \ldots$ or $u_n = \frac{1}{6}(5n^3 + 3n^2 - 32n + 30)$.

(iii) $1, 4, 18, 48, 99, \ldots$ or $u_n = \frac{1}{6}(5n^3 + 3n^2 - 26n + 24)$.

Many mathematical statements assert that a property is true for all positive integers. Bona [2] and Rosen [9] say that mathematical induction is a very important proof technique that can be used to prove the statement of integers. Using the induction mathematics and we can derive the following theorem.

**Theorem 2.1** For each sequence obtained from the corner heptagonal pyramid $1, 2, 9, 27, 61, \ldots$ the following $n$-term formula applies:

$$u_n = \frac{5}{6}n^3 - 2n^2 + \frac{7}{6}n + 1.$$ 

**Proof.** The theorem is proved using mathematical induction.

$$1 + 2 + 9 + 27 + 61 + \cdots + \frac{5n^3 - 12n^2 + 7n + 6}{6} = \frac{5n^4 - 6n^3 - 5n^2 + 30n}{24}.$$
(a) We observe that when $n=1$, it is true that $s(1) = 1$

\[
s(1) = \frac{1}{24}(5n^3 - 6n^2 - 5n + 30) = \frac{5k^4 - 6k^3 - 5k^2 + 30k}{24}
\]

(b) For the inductive hypothesis we assume that $n = k$ holds for an arbitrary positive integer $k$. That is we assume that,

\[
1 + 2 + 9 + 27 + 61 + \cdots + \frac{5k^3 - 12k^2 + 7k + 6}{6} = \frac{5k^4 - 6k^3 - 5k^2 + 30k}{24}.
\]

Under this assumption, it must be shown that $s(k + 1)$ is true, namely

\[
1 + 2 + 9 + 27 + 61 + \cdots + \frac{1}{6}(5k^3 - 12k^2 + 7k + 6) + \frac{1}{6}(5(k + 1)^3 - 12(k + 1)^2 + 7(k + 1) + 6) = \frac{1}{24}(5k^4 + 14k^3 + 7k^2 + 22k + 24).
\]

The true statement for $n = k + 1$. Thus it can be concluded that the given statement is true.

**Theorem 2.2** For each sequence obtained from the corners heptagonal pyramid $1, 3, 16, 45, 95, \cdots$ the following $n$-term formula applies:

\[
u_n = \frac{1}{6}(5n^3 + 3n^2 - 32n + 30).
\]

**Proof.** The theorem is proved using mathematical induction.

\[
1 + 3 + 16 + 45 + \cdots + \frac{5n^3 + 3n^2 - 32n + 30}{6} = \frac{5n^4 + 14n^3 - 53n^2 + 58n}{24}.
\]

(a) We observe that when $n = 1$, it is true that $s(1) = 1$

\[
s(1) = \frac{1}{24}(5n^4 + 14n^3 - 53n^2 + 58n) = \frac{5k^4 + 14k^3 - 53k^2 + 58k}{24}.
\]

(b) For the inductive hypothesis we assume that $n = k$ holds for an arbitrary positive integer $k$. That is we assume that:

\[
1 + 3 + 16 + 45 + 95 + \cdots + \frac{5k^3 + 3k^2 - 32k + 30}{6} = \frac{5k^4 + 14k^3 - 53k^2 + 58k}{24}.
\]

Under this assumption, it must be shown that $s(k + 1)$ is true, namely

\[
1 + 3 + 16 + 45 + \cdots + \frac{1}{6}(5k^3 + 3k^2 - 32k + 30) + \frac{1}{6}(5(k + 1)^3 + 14(k + 1)^2 + 7(k + 1) + 6) = \frac{1}{24}(5k^4 + 34k^3 + 19k^2 + 14k + 24).
\]

The true statement for $n = k + 1$. Thus it can be concluded that the given statement is true.
Theorem 2.3  For each sequence obtained from the corners heptagonal pyramid \(1, 4, 18, 48, 99, \cdots \) the following \(n\)-term formula applies:

\[ u_n = \frac{1}{6}(5n^3 + 3n^2 - 26n + 24). \]

Proof. The theorem is proved using mathematical induction.

\[ 1 + 4 + 18 + 48 + \cdots + \frac{5n^3 + 3n^2 - 26n + 24}{6} = \frac{5n^4 + 14n^3 - 41n^2 + 46n}{24}. \]

(a) We observe that when \(n = 1\), it is true that \(s(1) = 1\)

\[ s(1) = \frac{1}{24}(5n^4 + 14n^3 - 41n^2 + 46n) \]

\[ s(1) = 1. \]

(b) For the inductive hypothesis we assume that \(n = k\) holds for an arbitrary positive integer \(k\). That is we assume that:

\[ 1 + 4 + 18 + 48 + \cdots + \frac{5k^3 + 3k^2 - 26k + 24}{6} = \frac{5k^4 + 14k^3 - 41k^2 + 46k}{24} \]

Under this assumption, it must be shown that \(s(k + 1)\) is true, namely that:

\[ 1 + 4 + 18 + 48 + 99 + \cdots + \frac{5(k+1)^3 + 3(k+1)^2 - 26(k+1) + 24}{6} = \frac{5(k+1)^4 + 14(k+1)^3 - 41(k+1)^2 + 46(k+1)}{24} \]

The true statement for \(n = k + 1\). Thus it can be concluded that the given statement is true.

Remarks: Basically there are seven sequences obtained from the corner of heptagonal pyramids, namely:

(iv) \(1, 5, 20, 51, 103, \cdots \) or \( u_n = \frac{1}{6}(5n^3 + 3n^2 - 20n + 18) \).

(v) \(1, 6, 22, 54, 107, \cdots \) or \( u_n = \frac{1}{6}(5n^3 + 3n^2 - 14n + 12) \).

(vi) \(1, 7, 24, 57, 111, \cdots \) or \( u_n = \frac{1}{6}(5n^3 + 3n^2 - 8n + 6) \).

(vii) \(1, 8, 26, 60, 115, \cdots \) or \( u_n = \frac{1}{6}(5n^3 + 3n^2 - 2n) \).
3 Conclusion

In this paper the corner sequences of the heptagonal pyramid is obtained by using a three-dimensional sketch. For pyramids with many facets they can be obtained in the same way.

References


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