On the Function $\omega(n)$

Rafael Jakimczuk

División Matemática, Universidad Nacional de Luján
Buenos Aires, Argentina

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Abstract

Let $\omega(n)$ be the number of distinct primes in the prime factorization of $n$. In this article we prove that the numbers $n$ such that $\omega(n)$ is even have density $\frac{1}{2}$. We also prove another theorems on quadratfrei numbers.

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1 Introduction and Main Results

In this article (see [1, Chapter XXII]) $\omega(n)$ denotes the number of distinct primes in the prime factorization of $n$ and $\Omega(n)$ denotes the total number of primes in the prime factorization of $n$. That is, if the prime factorization of $n$ is $q_1^{r_1} \cdots q_s^{r_s}$, where the $q_i$ ($i = 1, \ldots, s$) ($s \geq 1$) are the different primes and the $r_i$ ($i = 1, \ldots, s$) are the multiplicities or exponents, we have $\omega(n) = s$ and $\Omega(n) = r_1 + \cdots + r_s$.

A quadratfrei number or squarefree number is a product of distinct primes, that is, a number such that its prime factorization is of the form $q_1 \cdots q_s$ where the $q_i$ ($i = 1, \ldots, s$) are the distinct primes in the prime factorization. Let $Q(x)$ be the number of squarefree not exceeding $x$. It is well-known (see [1, Chapter XVIII]) these numbers have positive density $\frac{6}{\pi^2}$. That is,

\[ Q(x) = \frac{6}{\pi^2} x + o(x) \]  

(1)
Let $Q_p(x)$ be the number of squarefree not exceeding $x$ with an even number of prime factors and let $Q_i(x)$ be the number of squarefree not exceeding $x$ with a odd number of prime factors. We have

$$Q(x) = Q_p(x) + Q_i(x) = \frac{6}{\pi^2} x + o(x)$$

On the other hand, as it is well-known, the prime number theorem is equivalent to

$$M(x) = \sum_{n \leq x} \mu(n) = o(x) = Q_p(x) - Q_i(x)$$

where $M(x)$ is the well-known Mertens’s function and $\mu(n)$ is the well-known Möbius function.

Equations (2) and (3) give

$$Q_i(x) = \frac{1}{2} \frac{6}{\pi^2} x + o(x)$$

$$Q_p(x) = \frac{1}{2} \frac{6}{\pi^2} x + o(x)$$

These two equations are also another equivalent establishment of the prime number theorem.

Let $\Omega_p(x)$ be the number of positive integers $n$ not exceeding $x$ such that $\Omega(n)$ is even and $\Omega_i(x)$ the number of positive integers $n$ not exceeding $x$ such that $\Omega(n)$ is odd. The following theorem is well-known (it is a consequence of the prime number theorem). For sake of completeness we give a simple proof. The author do not know if this simple proof is well-known.

**Theorem 1.1** The following two asymptotic formulae hold

$$\Omega_i(x) = \frac{1}{2} x + o(x)$$

$$\Omega_p(x) = \frac{1}{2} x + o(x)$$

Proof. Let us consider the positive integers not exceeding $x$ such that $n^2$ is their greatest square factor. Since $\Omega(n^2)$ is even, we have (see equation (5))

$$\Omega_p(x) = \sum_{n=1}^{N} \frac{Q_p \left( \frac{x}{n^2} \right)}{n^2} + F(x) = \sum_{n=1}^{N} \left( \frac{1}{2} \frac{6}{\pi^2} \frac{x}{n^2} + o(x) \right) + F(x)$$

$$= \frac{1}{2} \frac{6}{\pi^2} x \sum_{n=1}^{N} \frac{1}{n^2} + o(x) + F(x) = \frac{1}{2} x - \frac{1}{2} \frac{6}{\pi^2} x \sum_{n>N} \frac{1}{n^2} + o(x) + F(x)$$

$$+ F(x)$$

$$= \frac{1}{2} x + o(x)$$
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Since, as it is well-known, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Let $\epsilon > 0$, we shall choose $N$ such that $\sum_{N<n} \frac{1}{n^2} \leq \epsilon$. Therefore we have

$$0 \leq F(x) \leq \sum_{N<n} \frac{x}{n^2} \leq \epsilon x \tag{9}$$

Equations (8) and (9) give

$$\left| \frac{\Omega_p(x)}{x} - \frac{1}{2} \right| \leq 2\epsilon + \frac{1}{2} \frac{6}{\pi^2} \epsilon \leq 3\epsilon$$

That is, equation (7). Since $\epsilon$ can be arbitrarily small. Equation (6) can be proved in the same way or by difference. The theorem is proved.

Corollary 1.2 The following asymptotic formula holds.

$$\Omega_p(x) - \Omega_i(x) = \sum_{n \leq x} (-1)^{\Omega(n)} = o(x) \tag{10}$$

It is well known that equations (6) and (7) or the equivalent equation (10) imply the prime number theorem (see, for example, Theorem 1.7 in this paper for a simple proof). Therefore equations (6) and (7) or the equivalent equation (10) are equivalent to the prime number theorem, as it is well-known.

Let $\omega_p(x)$ be the number of positive integers $n$ not exceeding $x$ such that $\omega(n)$ is even and $\omega_i(x)$ the number of positive integers $n$ not exceeding $x$ such that $\omega(n)$ is odd. As a consequence of the prime number theorem we shall prove a similar theorem as Theorem 1.1. However the proof is not so simple as the proof of Theorem 1.1. The rest of the paper is dedicated to the proof of a similar theorem as Theorem 1.1 for the function $\omega(n)$.

We shall need the following well-known theorem.

Theorem 1.3 (Inclusion-exclusion principle) Let $S$ be a set of $N$ distinct elements, and let $S_1, \ldots, S_r$ be arbitrary subsets of $S$ containing $N_1, \ldots, N_r$ elements, respectively. For $1 \leq i < j < \ldots < l \leq r$, let $S_{ij\ldots l}$ be the intersection of $S_i, S_j, \ldots, S_l$ and let $N_{ij\ldots l}$ be the number of elements of $S_{ij\ldots l}$. Then the number $K$ of elements of $S$ not in any of $S_1, \ldots, S_r$ is

$$K = N - \sum_{1 \leq i \leq r} N_i + \sum_{1 \leq i < j \leq r} N_{ij} - \sum_{1 \leq i < j < k \leq r} N_{ijk} + \ldots + (-1)^r N_{12\ldots r}$$

Proof. See, for example, [3] (page 84) or [1] (page 233).

Let $q_1, \ldots, q_s$ be $s$ distinct primes. Let $P_{q_1\ldots q_s}(x)$ be the number of positive integers $n$ not exceeding $x$ such that $\Omega(n)$ is even and such that in their prime factorization appear the primes $q_1, \ldots, q_s$ with odd multiplicity. On the other hand, let $I_{q_1\ldots q_s}(x)$ be the number of positive integers $n$ not exceeding $x$ such that $\Omega(n)$ is odd and such that in their prime factorization appear the primes $q_1, \ldots, q_s$ with odd multiplicity. We have the following theorem.
Theorem 1.4 The following asymptotic formulae hold.
\[ P_{q_1 \cdots q_s}(x) = \frac{1}{2} \prod_{i=1}^{s} \frac{1}{q_i + 1} x + o(x) \]  
(11)
\[ I_{q_1 \cdots q_s}(x) = \frac{1}{2} \prod_{i=1}^{s} \frac{1}{q_i + 1} x + o(x) \]  
(12)

Proof. The number of positive integers \( n \) not exceeding \( x \) such that \( \Omega(n) \) is even and relatively prime with \( q_1 \cdots q_s \) will be (inclusion exclusion principle)
\[ \Omega_p(x) - \sum_{1 \leq i \leq s} \Omega_i \left( \frac{x}{q_i} \right) + \sum_{1 \leq i < j \leq s} \Omega_i \left( \frac{x}{q_i q_j} \right) - \cdots = \frac{1}{2} x - \sum_{1 \leq i \leq s} \frac{1}{2} \frac{x}{q_i} \]
\[ + \sum_{1 \leq i < j \leq s} \frac{1}{2} \frac{x}{q_i q_j} - \cdots + o(x) = \frac{1}{2} \prod_{i=1}^{s} \left( 1 - \frac{1}{q_i} \right) x + o(x) \]  
(13)

Analogously, the number of positive integers \( n \) not exceeding \( x \) such that \( \Omega(n) \) is odd and relatively prime with \( q_1 \cdots q_s \) will be
\[ \frac{1}{2} \prod_{i=1}^{s} \left( 1 - \frac{1}{q_i} \right) x + o(x) \]  
(14)

Let us consider the numbers whose prime factorization is of the form \( q_1^{r_1} \cdots q_s^{r_s} \) where \( r_i \) (\( i = 1, \ldots, s \)) is odd. We have
\[ \frac{1}{2} \prod_{i=1}^{s} \left( 1 - \frac{1}{q_i} \right) \sum_{q_1^{r_1} \cdots q_s^{r_s} \leq A} \frac{1}{q_1^{r_1} \cdots q_s^{r_s}} \]
\[ = \frac{1}{2} \prod_{i=1}^{s} \left( 1 - \frac{1}{q_i} \right) \left( \frac{1}{q_1 + 1} + \frac{1}{q_1^2} + \cdots \right) \left( \frac{1}{q_s + 1} + \frac{1}{q_s^2} + \cdots \right) \]
\[ = \frac{1}{2} \prod_{i=1}^{s} \frac{1}{q_i + 1} \]  
(15)

Let \( \epsilon > 0 \). We shall choose the number \( A \) such that
\[ \sum_{q_1^{r_1} \cdots q_s^{r_s} > A} \frac{1}{q_1^{r_1} \cdots q_s^{r_s}} \leq \epsilon \]  
(16)

Therefore we have (see (13), (14), (15) and (16))
\[ P_{q_1 \cdots q_s}(x) = \sum_{q_1^{r_1} \cdots q_s^{r_s} \leq A} \left( \frac{1}{2} \prod_{i=1}^{s} \left( 1 - \frac{1}{q_i} \right) \frac{x}{q_1^{r_1} \cdots q_s^{r_s}} + o(x) \right) + F(x) \]
\[ = x \frac{1}{2} \prod_{i=1}^{s} \left( 1 - \frac{1}{q_i} \right) \sum_{q_1^{r_1} \cdots q_s^{r_s} \leq A} \frac{1}{q_1^{r_1} \cdots q_s^{r_s}} + o(x) + F(x) \]
\[ = \frac{1}{2} \prod_{i=1}^{s} \frac{1}{q_i + 1} x - x \frac{1}{2} \prod_{i=1}^{s} \left( 1 - \frac{1}{q_i} \right) \sum_{q_1^{r_1} \cdots q_s^{r_s} > A} \frac{1}{q_1^{r_1} \cdots q_s^{r_s}} + o(x) + F(x) \]  
(17)
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where (see (16))

$$0 \leq F(x) \leq \sum_{q_1 \cdots q_s > A} \frac{x}{q_1 \cdots q_s} \leq \epsilon x$$

Equations (16), (17) and (18) give

$$\left| \frac{P_{q_1 \cdots q_s}(x)}{x} - \frac{1}{2} \prod_{i=1}^{s} \frac{1}{q_i} \right| \leq \frac{1}{2} \prod_{i=1}^{s} \left( 1 - \frac{1}{q_i} \right) \epsilon + \epsilon + \epsilon \leq 3 \epsilon$$

That is (11), since $\epsilon$ can be arbitrarily small. The proof of equation (12) is the same. The theorem is proved.

We shall need the following two well-known theorems.

**Theorem 1.5** (The second Möbius inversion formula) Let $f(x)$ and $g(x)$ be functions defined for $x \geq 1$. If

$$g(x) = \sum_{n \leq x} f \left( \frac{x}{n} \right) \quad (x \geq 1)$$

then

$$f(x) = \sum_{n \leq x} \mu(n) g \left( \frac{x}{n} \right) \quad (x \geq 1)$$

where $\mu(n)$ is the Möbius function.

Proof. See, for example, [1, Chapter XVI, Theorem 268].

**Theorem 1.6** The following formula holds

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2}$$

Proof. See, for example, [1, Chapter XVII, Theorem 287 and page 245].

Let $(MP)_{q_1 \cdots q_s}(x)$ be the number of squarefree $n$ not exceeding $x$ multiple of $q_1 \cdots q_s$ such that $\Omega(n) = \omega(n)$ is even. On the other hand, let $(MI)_{q_1 \cdots q_s}(x)$ be the number of squarefree $n$ not exceeding $x$ multiple of $q_1 \cdots q_s$ such that $\Omega(n) = \omega(n)$ is odd. We have the following theorem.

**Theorem 1.7** The following asymptotic formulae hold.

$$(MP)_{q_1 \cdots q_s}(x) = \frac{1}{2} \frac{6}{\pi^2} \prod_{i=1}^{s} \frac{1}{q_i} x + o(x)$$

$$(MI)_{q_1 \cdots q_s}(x) = \frac{1}{2} \frac{6}{\pi^2} \prod_{i=1}^{s} \frac{1}{q_i} x + o(x)$$
Proof. We have (see Theorem 1.4 and [1, Chapter XVIII, Theorem 333]).

\[ P_{q_1 \cdots q_s}(y^2) = cy^2 + o(y^2) = cy^2 + f(y^2)y^2 = \sum_{d \leq y} (MP)_{q_1 \cdots q_s} \left( \left( \frac{y}{d} \right)^2 \right) \]

where for sake of simplicity we put \( c = \frac{1}{2} \prod_{i=1}^{s} \frac{1}{q_i + 1} \). Besides \( \lim_{x \to \infty} f(x) = 0 \) and \( |f(x)| < M \). By Theorem 1.5 and Theorem 1.6 we have

\[ (MP)_{q_1 \cdots q_s}(y^2) = \sum_{d \leq y} \mu(d) \left( c \left( \frac{y}{d} \right)^2 + f \left( \left( \frac{y}{d} \right)^2 \right) \frac{y^2}{d^2} \right) = y^2 \sum_{d \leq y} \frac{\mu(d)}{d^2} \]

If we put \( y^2 = x \) then we obtain equation (19). Equation (20) can be proved in the same way. Note that

\[ y^2 c \sum_{d > y} \frac{\mu(d)}{d^2} = O(y) \]

and

\[ y^2 \sum_{d \leq y} f \left( \left( \frac{y}{d} \right)^2 \right) \frac{\mu(d)}{d^2} = y^2 \sum_{d \leq \sqrt{y}} f \left( \left( \frac{y}{d} \right)^2 \right) \frac{\mu(d)}{d^2} \]

Since for all \( \epsilon > 0 \) we have

\[ \left| \sum_{d \leq \sqrt{y}} f \left( \left( \frac{y}{d} \right)^2 \right) \frac{\mu(d)}{d^2} \right| \leq \sum_{d \leq \sqrt{y}} \left| f \left( \left( \frac{y}{d} \right)^2 \right) \right| \frac{1}{d^2} \]

\[ + \sum_{\sqrt{y} < d \leq y} \left| f \left( \left( \frac{y}{d} \right)^2 \right) \right| \frac{1}{d^2} \leq \pi^2 \frac{\epsilon}{6} + M o(1) \leq 3 \epsilon \]

The theorem is proved.

Let \((RPP)_{q_1 \cdots q_s}(x)\) be the number of squarefree \( n \) not exceeding \( x \), relatively prime to \( q_1 \cdots q_s \) and such that \( \Omega(n) = \omega(n) \) is even. Let \((RPI)_{q_1 \cdots q_s}(x)\) be the number of squarefree \( n \) not exceeding \( x \), relatively prime to \( q_1 \cdots q_s \) and such that \( \Omega(n) = \omega(n) \) is odd. The following theorem holds.
Theorem 1.8  The following asymptotic formulae hold.

\[(RPP)_{q_1 \cdots q_s}(x) = \frac{1}{2} \frac{6}{\pi^2} \prod_{i=1}^{s} \frac{q_i}{q_i + 1} x + o(x)\]  
(21)

\[(RPI)_{q_1 \cdots q_s}(x) = \frac{1}{2} \frac{6}{\pi^2} \prod_{i=1}^{s} \frac{q_i}{q_i + 1} x + o(x)\]  
(22)

Proof. By the inclusion-exclusion principle, Theorem 1.7 and (5) we have

\[(RPP)_{q_1 \cdots q_s}(x) = \frac{1}{2} \frac{6}{\pi^2} x + o(x) - \sum_{1 \leq i \leq s} \left( \frac{1}{2} \frac{6}{\pi^2} \frac{1}{q_i + 1} x + o(x) \right)
+ \sum_{1 \leq i < j \leq s} \left( \frac{1}{2} \frac{6}{\pi^2} \frac{1}{(q_i + 1)(q_j + 1)} x + o(x) \right) - \cdots
= \frac{1}{2} \frac{6}{\pi^2} x \prod_{i=1}^{s} \left( 1 - \frac{1}{q_i + 1} \right) + o(x)\]

That is, equation (21). Equation (22) can be proved in the same way. The theorem is proved.

A number is powerful or squareful if all the distinct primes in its prime factorization have multiplicity (or exponent) greater than 1. That is, the number \(q_1^{r_1} \cdots q_s^{r_s}\) is squareful if \(r_i \geq 2 (i = 1, \ldots, s) \ (s \geq 1)\).

Theorem 1.9  We have

\[\sum_{q_1^{r_1} \cdots q_s^{r_s}} \frac{6}{\pi^2 (q_1 + 1) \cdots (q_s + 1) q_1^{r_1} \cdots q_s^{r_s}} = 1 - \frac{6}{\pi^2}\]

where the sum run on all squareful numbers \(q_1^{r_1} \cdots q_s^{r_s}\).

Proof. We have

\[\sum_{q_1^{r_1} \cdots q_s^{r_s}} \frac{6}{\pi^2 (q_1 + 1) \cdots (q_s + 1) q_1^{r_1} \cdots q_s^{r_s}} = \frac{6}{\pi^2} \prod_p \left( 1 + \frac{1}{p + 1} \right) - \frac{6}{\pi^2} = \frac{6}{\pi^2} \prod_p \left( 1 + \frac{1}{p + 1} \right) - \frac{6}{\pi^2} = 1 - \frac{6}{\pi^2}\]

The theorem is proved.

Now, we can prove our main theorem.
Theorem 1.10 The following asymptotic formulae hold.

\[ \omega_p(x) = \frac{1}{2}x + o(x) \quad (23) \]

\[ \omega_i(x) = \frac{1}{2}x + o(x) \quad (24) \]

Proof. In the proof of this theorem \( q_1^{r_1} \cdots q_s^{r_s} \) denotes a squareful number. We have (see Theorem 1.8, Theorem 1.9 and (5))

\[ \omega_p(x) = \frac{6}{\pi^2} x + o(x) \]

\[ \omega_i(x) = \frac{1}{2}x + o(x) \quad (25) \]

It is well-known that the series of the reciprocal of the squareful numbers converges and that the squareful numbers have zero density, that is, the number of squareful numbers not exceeding \( x \) is \( o(x) \). Let \( \epsilon > 0 \). Consequently we choose \( A \) such that

\[ \sum_{q_1^{r_1} \cdots q_s^{r_s} > A} \frac{1}{q_1^{r_1} \cdots q_s^{r_s}} \leq \epsilon \]

and hence

\[ \sum_{q_1^{r_1} \cdots q_s^{r_s} > A} \left( \frac{6}{\pi^2} \frac{q_1 \cdots q_s}{(q_1+1) \cdots (q_s+1)} \frac{1}{q_1^{r_1} \cdots q_s^{r_s}} \right) \leq \epsilon \quad (26) \]

and

\[ 0 \leq F(x) \leq \sum_{q_1^{r_1} \cdots q_s^{r_s} > A} \frac{x}{q_1^{r_1} \cdots q_s^{r_s}} \leq \epsilon x \quad (27) \]

Finally, equations (25), (26) and (27) give

\[ \left| \frac{\omega_p(x)}{x} - \frac{1}{2} \right| \leq \epsilon + \epsilon + \frac{1}{2} \epsilon \leq 3 \epsilon \]

That is, equation (23), since \( \epsilon \) can be arbitrarily small. The proof of equation (24) is the same or by difference. The theorem is proved.

In the following theorem we prove that the prime number theorem is equivalent to a proposition on a number of squarefree of positive density arbitrarily small.
Theorem 1.11 Let \( p_n \) be the \( n \)-th prime. Let us consider the first \( n \) primes. The prime number theorem is equivalent to the following two formulae.

\[
(RPP)_{p_1 \cdots p_n}(x) = \frac{1}{2} \frac{6}{\pi^2} \prod_{i=1}^{n} \frac{p_i}{p_i + 1} x + o(x) = \frac{1}{2} \frac{6}{\pi^2} \prod_{i=1}^{n} \left( \frac{1}{p_i} + \frac{1}{p_i} \right) x + o(x) \quad (28)
\]

\[
(RPI)_{p_1 \cdots p_n}(x) = \frac{1}{2} \frac{6}{\pi^2} \prod_{i=1}^{n} \frac{p_i}{p_i + 1} x + o(x) = \frac{1}{2} \frac{6}{\pi^2} \prod_{i=1}^{n} \left( \frac{1}{p_i} + \frac{1}{p_i} \right) x + o(x) \quad (29)
\]

Proof. Equations (28) and (29) are a particular case of Theorem 1.8 and Theorem 1.8 is a consequence of the prime number theorem (see the proof). Reciprocally, suppose that equations (28) and (29) holds, then

\[
Q_p(x) = \frac{1}{2} \frac{6}{\pi^2} \prod_{i=1}^{n} \frac{p_i}{p_i + 1} x + o(x) + \sum_{i=1}^{n} \left( \frac{1}{2} \frac{6}{\pi^2} \prod_{i=1}^{n} \frac{p_i}{p_i + 1} x + o(x) \right)
\]

\[
+ \sum_{1 \leq i < j \leq n} \left( \frac{1}{2} \frac{6}{\pi^2} \prod_{i=1}^{n} \frac{p_i}{p_i + 1} x + o(x) \right) + \cdots = \frac{1}{2} \frac{6}{\pi^2} x + o(x)
\]

That is, equation (5).

In the same way we obtain (4) or by difference. On the other hand, we have \( \lim_{n \to \infty} \left( 1 + \frac{1}{p_i} \right) = \infty \). The theorem is proved.

Theorem 1.12 Let \( n' \) be a squarefree relatively prime to \( p_1 \cdots p_n \). Then, the prime number theorem is equivalent to the following establishment

\[
\sum_{n' \leq x} \mu(n') = o(x) \quad (30)
\]

Compare with equation (3).

Proof. The prime number theorem implies (30) since

\[
\sum_{n' \leq x} \mu(n') = (RPP)_{p_1 \cdots p_n}(x) - (RPP)_{p_1 \cdots p_n}(x) = o(x) \quad (31)
\]

(see Theorem 1.11). Reciprocally, suppose that (30) holds, without the use of the prime number theorem can be proved (see [2]) that

\[
(RPP)_{p_1 \cdots p_n}(x) + (RPP)_{p_1 \cdots p_n}(x) = \frac{6}{\pi^2} \prod_{i=1}^{n} \frac{p_i}{p_i + 1} x + o(x) \quad (32)
\]

From (31) and (32) we obtain (28) and (29). Note that the \( n' \) have positive density arbitrarily small if \( n \) is large (Theorem 1.11). The theorem is proved.

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