Composite Numbers IV with Applications to 
the Normal Order of an Arithmetical Function, 
the Kernel Function and the ABC Conjecture

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Abstract

We prove a theorem on the distribution of certain sets of composite numbers. The proof use as main lemma a result of the author on the distribution of certain sets of quadratfrei numbers. After, we define the arithmetical function $a(n)$, where $a(n)$ is the number of primes with multiplicity (or exponent) 1 in the prime factorization of $n$ and prove using the previous results on composite numbers that its normal order is $\log \log n$. Also, using the previous results on composite numbers, we obtain a theorem on the kernel function and as corollary we obtain that the ABC conjecture holds for almost all C.

Mathematics Subject Classification: 11A99, 11B99

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1 Introduction and Preliminary Notes

A quadratfrei number or squarefree number is a product of distinct primes, that is, a number such that its prime factorization is of the form $q_1 \cdots q_s$ where the $q_i$ ($i = 1, \ldots, s$) are the distinct primes in the prime factorization. The set of the squarefree numbers will be denoted $C_0$. Let $Q_0(x)$ be the number
of squarefree not exceeding \( x \). It is well-known (see [1, Chapter XVIII]) these numbers have positive density \( \rho_0 = \frac{6}{\pi^2} \). That is,

\[
Q_0(x) = \rho_0 x + o(x) = \frac{6}{\pi^2} x + o(x)
\] (1)

Let us consider the set of squarefree relatively prime to the squarefree fixed \( q_1 \cdots q_s \). The number of these squarefree not exceeding \( x \) will be denoted \( Q_{q_1, \ldots, q_s}(x) \). It is well-known (see [2]) that

\[
Q_{q_1, \ldots, q_s}(x) = \frac{6}{\pi^2} \frac{q_1 \cdots q_s}{(q_1 + 1) \cdots (q_s + 1)} x + o(x)
\] (2)

A number such that all primes in its prime factorization has multiplicity (exponent) greater than 1 is called squareful or powerful number. It is well-known that the series of the reciprocal of the squareful numbers converges.

In this article we study the numbers such that their prime factorization is of the form \( q_1 \cdots q_t q_{t+1}^{r_{t+1}} \cdots q_{t+s}^{r_{t+s}} \) where \( q_1, \ldots, q_{t+s} \) are the distinct primes in the prime factorization, \( t \geq 1 \) is variable, \( s \geq 1 \) is fixed and the \( s \) variable exponents \( r_{t+1}, \ldots, r_{t+s} \) are greater than 1. Therefore, these numbers have a variable squarefree part \( q = q_1 \cdots q_t \) and a variable powerful part \( q_{t+1}^{r_{t+1}} \cdots q_{t+s}^{r_{t+s}} \) with a number fixed \( s \) of distinct primes We shall denote this variable powerful part in the form \( q_1^{r_1} \cdots q_s^{r_s} \). Consequently we shall denote these numbers \( q_1 \cdots q_t q_{t+1}^{r_{t+1}} \cdots q_{t+s}^{r_{t+s}} \) in the compact form \( q q_1^{r_1} \cdots q_s^{r_s} \) where \( q \) is the squarefree part and \( q_1^{r_1} \cdots q_s^{r_s} \) is the powerful part. The set of these numbers will be denoted \( C_s \). The number of these numbers not exceeding \( x \) will be denoted \( Q_s(x) \). We shall prove that these numbers have positive density \( \rho_s \).

The following well-known formula will be used (see [1, Chapter XXII])

\[
\sum_{p \leq x} \frac{1}{p} = \log \log x + M + o(1)
\] (3)

where \( M \) is called Mertens’s constant and \( p \) denotes a positive prime.

2 Main Results

**Theorem 2.1** Let \( s \) be an arbitrary but fixed positive integer. The following formula holds

\[
Q_s(x) = \rho_s x + o(x)
\] (4)

where

\[
\rho_s = \frac{6}{\pi^2} \sum_{q_1 \cdots q_s} \frac{1}{(q_1^2 - 1) \cdots (q_s^2 - 1)}
\] (5)

The notation \( \sum_{q_1 \cdots q_s} \) mean that the sum run on all squarefree numbers \( q_1 \cdots q_s \) with exactly \( s \) distinct prime factors.
Proof. We put
\[ \rho_s = \frac{6}{\pi^2} \sum_{q_1^{r_1} \ldots q_s^{r_s}} \frac{q_1 \ldots q_s}{(q_1 + 1) \ldots (q_s + 1) q_1^{r_1} \ldots q_s^{r_s}} \tag{6} \]
where \( \sum_{q_1^{r_1} \ldots q_s^{r_s}} \) mean that the sum run on all powerful numbers \( q_1^{r_1} \ldots q_s^{r_s} \) with exactly \( s \) distinct prime factors. Note that the series converges, since the series of the reciprocal of the powerful numbers converges (see the introduction).

Let \( \epsilon > 0 \). We choose the powerful number \( A \) with exactly \( s \) distinct prime factors such that
\[ \sum_{q_1^{r_1} \ldots q_s^{r_s} > A} \frac{1}{q_1^{r_1} \ldots q_s^{r_s}} < \epsilon \tag{7} \]
\[ \sum_{q_1^{r_1} \ldots q_s^{r_s} > A} \frac{q_1 \ldots q_s}{(q_1 + 1) \ldots (q_s + 1) q_1^{r_1} \ldots q_s^{r_s}} < \frac{\epsilon^2}{6} \tag{8} \]

Let us consider a fixed powerful number \( q_1^{r_1} \ldots q_s^{r_s} \). The number of numbers of the form \( qq_1^{r_1} \ldots q_s^{r_s} \) (see the introduction) not exceeding \( x \) will be denoted \( Q_{qq_1^{r_1} \ldots q_s^{r_s}}(x) \) and consequently (see (2)) we have
\[ Q_{qq_1^{r_1} \ldots q_s^{r_s}}(x) = \frac{6}{\pi^2} \frac{q_1 \ldots q_s}{(q_1 + 1) \ldots (q_s + 1) q_1^{r_1} \ldots q_s^{r_s}} \frac{x}{q_1^{r_1} \ldots q_s^{r_s}} + o(x) \tag{9} \]

Therefore (see (9) and (6))
\[ Q_s(x) = \sum_{q_1^{r_1} \ldots q_s^{r_s} \leq A} Q_{qq_1^{r_1} \ldots q_s^{r_s}}(x) + P(x) \]
\[ = \sum_{q_1^{r_1} \ldots q_s^{r_s} \leq A} \frac{6}{\pi^2} \frac{q_1 \ldots q_s}{(q_1 + 1) \ldots (q_s + 1) q_1^{r_1} \ldots q_s^{r_s}} \frac{x}{q_1^{r_1} \ldots q_s^{r_s}} + o(x) + P(x) \]
\[ = \rho_s x - \left( \frac{6}{\pi^2} \sum_{q_1^{r_1} \ldots q_s^{r_s} > A} \frac{q_1 \ldots q_s}{(q_1 + 1) \ldots (q_s + 1) q_1^{r_1} \ldots q_s^{r_s}} \frac{1}{q_1^{r_1} \ldots q_s^{r_s}} \right) x + o(x) + P(x) \tag{10} \]

Equation (10) gives
\[ \frac{Q_s(x)}{x} - \rho_s \]
\[ = - \left( \frac{6}{\pi^2} \sum_{q_1^{r_1} \ldots q_s^{r_s} > A} \frac{q_1 \ldots q_s}{(q_1 + 1) \ldots (q_s + 1) q_1^{r_1} \ldots q_s^{r_s}} \frac{1}{q_1^{r_1} \ldots q_s^{r_s}} \right) x + o(1) + \frac{P(x)}{x} \tag{11} \]
Now, we have (see (7))
\[ 0 \leq P(x) \leq \sum_{q_1^{r_1} \cdots q_s^{r_s} > A} \frac{x}{q_1^{r_1} \cdots q_s^{r_s}} \leq x \sum_{q_1^{r_1} \cdots q_s^{r_s} > A} \frac{1}{q_1^{r_1} \cdots q_s^{r_s}} < \varepsilon x \]  
(12)

since \( \lfloor \frac{x}{a} \rfloor \) is the number of multiples of \( a \) not exceeding \( x \).

Equations (11), (8) and (2) give
\[ \left| \frac{Q_s(x)}{x} - \rho_s \right| < 3\varepsilon \quad (x \geq x_\varepsilon) \]  
(13)

and, since \( \varepsilon \) can be arbitrarily small, equation (13) gives equation (4).

If the distinct primes \( q_1, \ldots, q_s \) are fixed we have
\[ \sum \frac{q_1 \cdots q_s}{(q_1 + 1) \cdots (q_s + 1) q_1^{r_1} \cdots q_s^{r_s}} \frac{1}{q_1^{r_1} \cdots q_s^{r_s}} = \frac{q_1 \cdots q_s}{(q_1 + 1) \cdots (q_s + 1)} \left( \frac{1}{q_1^2} + \frac{1}{q_1^3} + \cdots \right) \cdots \left( \frac{1}{q_s^2} + \frac{1}{q_s^3} + \cdots \right) = \frac{1}{(q_1^2 - 1) \cdots (q_s^2 - 1)} \]  
(14)

Hence, equations (14) and (6) give equation (5). The theorem is proved.

Note that if (see the introduction) \( s_1 \neq s_2 \) then \( C_{s_1} \cap C_{s_2} \) is empty. On the other
hand \( \bigcup_{s=0}^\infty C_s = N - P \) where \( N \) denotes the set of the positive integers and \( P \) denotes
the set of powerful numbers. Consequently the \( C_s \) are a partition of the set \( N - P \). The density of the
set \( N - P \) is 1, since it is well-known that the density of the set of powerful numbers is zero. In the following theorem
we prove that the density 1 of the union \( \bigcup_{s=0}^\infty C_s \) is the sum of the densities of the sets \( C_s \).

**Theorem 2.2** We have
\[ \sum_{s=0}^\infty \rho_s = 1 \]  
(15)

Proof. We have (see (5) and (1))
\[ \sum_{s=0}^\infty \rho_s = \frac{6}{\pi^2} \prod_p \left( 1 + \frac{1}{p^2 - 1} \right) = \frac{6}{\pi^2} \prod_p \frac{1}{1 - 1/p^2} = 1 \]

The theorem is proved.

Let us consider the prime factorization of an positive integer \( n \). The number
of distinct primes in the prime factorization we denote $\omega(n)$ (see [1, Chapter XXII]), the number of primes in the prime factorization we denote $\Omega(n)$ (see [1, Chapter XXII]) and the number of primes with multiplicity (exponent) 1 we denote $a(n)$. The following formulae are well-known (see [1, Chapter XXII] and (3))

$$\sum_{n \leq x} \omega(n) = x \log \log x + Mx + o(x)$$

$$\sum_{n \leq x} \Omega(n) = x \log \log x + Bx + o(x)$$

where $B = M + \sum_p \frac{1}{p(p-1)}$. We also have

$$\sum_{n \leq x} a(n) = x \log \log x + Ax + o(x)$$

where $A = M - \sum_p \frac{1}{p^2}$. The proof is very simple, by equation (3) we have

$$\sum_{n \leq x} a(n) = \sum_{p \leq x} \left( \left\lfloor \frac{x}{p} \right\rfloor - \frac{x}{p^2} \right) = x \sum_{p \leq x} \frac{1}{p} - x \sum_{p \leq x} \frac{1}{p^2} + o(x)$$

$$= x \left( \log \log x + M + o(1) \right) - x \left( \sum_p \frac{1}{p^2} + o(1) \right) + o(x)$$

$$= x \log \log x + Ax + o(x)$$

Hence, the average order of the three functions $a(n)$, $\omega(n)$ and $\Omega(n)$ is log log $n$.

We say (see [1, Chapter XXII]) the normal order of $f(n)$ is $F(n)$ if and only if for all $\epsilon > 0$ there exists a set $S_\epsilon$ (depending of $\epsilon$) with density 1 such that

$$(1 - \epsilon) F(n) < f(n) < (1 + \epsilon) F(n) \quad (n \in S_\epsilon)$$

It is well-known (see [1, Chapter XXII]) that the normal order of $\omega(n)$ is log log $n$ and the normal order of $\Omega(n)$ is log log $n$. In the following theorem we prove that the normal order of $a(n)$ is also log log $n$.

**Theorem 2.3** The normal order of $a(n)$ is log log $n$.

Proof. Let $\epsilon > 0$. There exists a set $A_\epsilon$ of density 1 such that

$$(1 - \epsilon) \log \log n < \omega(n) < (1 + \epsilon) \log \log n \quad (n \in A_\epsilon)$$

(16)

since the normal order of $\omega(n)$ is log log $n$. Hence $\omega(n) \to \infty$ on the set $A_\epsilon$. 
Let us consider the subset $C_\epsilon \subseteq A_\epsilon$ such that

$$1 - \epsilon \leq \frac{a(n)}{\omega(n)} \leq 1$$  \hspace{1cm} (17)

Now, let us consider the set $C_s (s \geq 0)$ with density $\rho_s > 0$ (Theorem 2.1 and equation (1)). The set $C_s \cap A_\epsilon$ also has density $\rho_s$. Besides, we have $a(n) = \omega(n) - s$, hence

$$\frac{a(n)}{\omega(n)} = \frac{\omega(n) - s}{\omega(n)} \rightarrow 1 \quad (n \in (C_s \cap A_\epsilon))$$

since $\omega(n) \rightarrow \infty$. Consequently, except by a finite set $B_s \subseteq (C_s \cap A_\epsilon)$ with $n_s$ elements we have

$$1 - \epsilon \leq \frac{a(n)}{\omega(n)} \leq 1 \quad (n \in ((C_s \cap A_\epsilon) - B_s))$$

Let $\epsilon' > 0$. There exists $r$ such that (Theorem 2.2)

$$\sum_{s=0}^{r} \rho_s \geq 1 - \frac{\epsilon'}{2}$$  \hspace{1cm} (18)

Let $\alpha(N)$ be the number of elements in the set $C_\epsilon$ not exceeding $N$ and let $\beta_s(N)$ be the number of elements in the set $(C_s \cap A_\epsilon) - B_s$ not exceeding $N$, hence $\beta_s(N) = \rho_s N + o(N) - n_s$.

Now, we have

$$\frac{\alpha(N)}{N} \leq \frac{N}{N} = 1$$  \hspace{1cm} (19)

and (see (18))

$$\frac{\alpha(N)}{N} \geq \frac{\sum_{s=0}^{r} \beta_s(N)}{N} = \frac{\sum_{s=0}^{r} (\rho_s N + o(N) - n_s)}{N} = \left(\sum_{s=0}^{r} \rho_s\right) + o(1)$$

$$\geq 1 - \frac{\epsilon'}{2} - \frac{\epsilon'}{2} = 1 - \epsilon' \quad (N \geq N_{\epsilon'})$$  \hspace{1cm} (20)

Equations (19) and (20) give

$$\frac{\alpha(N)}{N} \rightarrow 1$$

since $\epsilon'$ can be arbitrarily small. That is, the subset $C_\epsilon$ has density 1. Now, in the subset $C_\epsilon$ we have (see (17) and (16))

$$(1 - 3\epsilon) \log \log n < (1 - \epsilon)^2 \log \log n < (1 - \epsilon)\omega(n) \leq a(n)$$

$$\leq \omega(n) < (1 + \epsilon) \log \log n < (1 + 3\epsilon) \log \log n$$
The theorem is proved.

Let \( b(n) \) be the number of primes in the prime factorization of \( n \) with multiplicity (or exponent) greater than 1. We have

\[
\sum_{n \leq x} b(n) = \sum_{p \leq x} \left\lfloor \frac{x}{p^2} \right\rfloor = Cx + o(x)
\]

where \( C = \sum_p \frac{1}{p^2} \).

Note that the intersection of a finite number of sets of density 1 is again a set of density 1 (the proof is very simple using the inclusion-exclusion principle and mathematical induction). Therefore in Theorem 2.3 we can take \( A_\epsilon \) of density 1 where the following two inequalities hold

\[
(1 - \epsilon) \log \log n < \omega(n) < (1 + \epsilon) \log \log n
\]

and consequently in the set \( C_\epsilon \) of density 1 (see the proof of theorem 2.3) hold (22), (23) and

\[
(1 - 3\epsilon) \log \log n < a(n) < (1 + 3\epsilon) \log \log n
\]

Note that

\[
a(n) + b(n) = \omega(n)
\]

Equations (25), (17) and (22) give us in the set \( C_\epsilon \) the inequality

\[
0 \leq b(n) < 2\epsilon \log \log n
\]

Consequently in the set \( C_\epsilon \) of density 1 hold (22), (23), (24) and (26).

**Theorem 2.4** Let \( \epsilon > 0 \) an arbitrary but fixed real number. Let \( A_\epsilon(x) \) the number of positive integers \( c \) not exceeding \( x \) such that

\[
c < u(c)^{1+\epsilon}
\]

Then \( \lim_{x \to \infty} \frac{A_\epsilon(x)}{x} = 1 \). That is, the set of numbers \( c \) that satisfy inequality (27) has density 1.

Proof. From the definition of \( A_\epsilon(x) \) we have the trivial inequality

\[
\frac{A_\epsilon(x)}{x} \leq 1
\]
On the other hand, we have (see (9))

\[
Q_{qq_1^{r_1}...q_s^{r_s}}(x) = \frac{6}{\pi^2} \frac{q_1 \cdot \cdots \cdot q_s}{(q_1 + 1) \cdots (q_s + 1)} x^{r_1} \cdots q_s^{r_s} + o(x) = \rho_{q_1^{r_1}...q_s^{r_s}} x + o(x) \quad (29)
\]

The inequality

\[
c = qq_1^{r_1} \cdots q_s^{r_s} < u(c)^{1+\epsilon} = (qq_1 \cdots q_s)^{1+\epsilon}
\]
is equivalent to the inequality

\[
\frac{q_1^{r_1} \cdots q_s^{r_s}}{(q_1 \cdots q_s)^{1+\epsilon}} < q^\epsilon
\]

and since the squarefrees \( q \to \infty \) this inequality holds for all \( q \) except for a finite number \( n_1^{r_1} \cdots n_s^{r_s} \) of \( q \).

We have (see (15))

\[
\frac{6}{\pi^2} + \sum_{q_1^{r_1} \cdots q_s^{r_s}} \rho_{q_1^{r_1} \cdots q_s^{r_s}} = 1
\]

Therefore if \( \alpha > 0 \) there exists \( B \) such that

\[
\frac{6}{\pi^2} + \sum_{q_1^{r_1} \cdots q_s^{r_s} < B} \rho_{q_1^{r_1} \cdots q_s^{r_s}} \geq 1 - \alpha \quad (30)
\]

Consequently, we have (see (1), (29) and (30))

\[
A_r(x) \geq \frac{6}{\pi^2} x + o(x) + \sum_{q_1^{r_1} \cdots q_s^{r_s} < B} \left( \rho_{q_1^{r_1} \cdots q_s^{r_s}} x + o(x) - n_{q_1^{r_1} \cdots q_s^{r_s}} \right) = \left( \frac{6}{\pi^2} + \sum_{q_1^{r_1} \cdots q_s^{r_s} < B} \rho_{q_1^{r_1} \cdots q_s^{r_s}} \right) x + o(x) \geq (1 - \alpha) x + o(x)
\]

\[
\geq (1 - 2\alpha)x \quad (31)
\]

Equations (28) and (31) give

\[
1 - 2\alpha \leq \frac{A_r(x)}{x} \leq 1 \quad (x \geq x_\alpha)
\]

and equation (32) gives the limit \( \lim_{x \to \infty} \frac{A_r(x)}{x} = 1 \), since \( \alpha \) can be arbitrarily small. The theorem is proved.

The abc conjecture establish that if \( a, b \) and \( c \) are positive and relatively prime integers which satisfy the equation \( a + b = c \) then for any \( \epsilon > 0 \), with finitely many exceptions, we have that

\[
c < (u(abc))^{1+\epsilon}
\]

Theorem 2.4 implies the following corollary
**Corollary 2.5** The set of numbers $c$ such that the abc conjecture holds has density 1.

Proof. Inequality (27) implies inequality (33). The corollary is proved.

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**References**


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