International Mathematical Forum, Vol. 13, 2018, no. 3, 117 - 122
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2018.71195

Extended Centerality in a Complex Banach Algebra

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Abstract

In this paper we define and study the extended center, the extended quasi center, the extended σ-quasi center and the extended ρ-quasi center of a complex Banach algebra, where we get some results that are similar to known results concerning center, quasi center, σ-quasi center and ρ-quasi center of a complex Banach algebra.

Keywords: Centrality, extended centrality, complex Banach algebra

1. Introduction

The purpose of this paper is to study extended centrality in a complex Banach algebra, where we get some result concerning these concepts. Most of these results and their proofs are similar to that for As'ad, C. Le Page and Rennison in [2], [6], [7], [8] and [9].

Throughout this paper all linear spaces and algebras are assumed to be defined over ¢ the field of complex numbers, A will denote a unital complex Banach algebra and the center of A is $Z(A) = \{ a \in A : ax = xa \text{ for all } x \in A \}$.

In [7] and [8] Rennison defined the set of all quasi central elements in A by Q(A)

= $\underset{k\geq 1}{\cup}$ Q(k, A), where Q(k, A) = { a∈A: ‖ x (λ - a) ‖ ≤ k ‖ (λ - a) x ‖ for all x∈A and all λ∈¢ }, and the set of all σ-quasi central elements in A by $\underset{k\geq 1}{\cup} Q_{\sigma}(\text{k}, \text{A})$, where $Q_{\sigma}(\text{k}, \text{A})$ = { a∈A : ‖ x (λ - a) ‖ ≤ k ‖ (λ - a) x ‖ for all x∈A and all λ∈ $\rho_A(a)$ }, then he show that Z(A) ⊆ Q(A) ⊆Q_{σ}(A). In [4] Hussein and As'ad defined the set of all ρ-quasi central elements in A by $Q_{\rho}(\text{A}) = \underset{k\geq 1}{\cup} Q_{\rho}(\text{k}, \text{A})$, where $Q_{\rho}(\text{k}, \text{A})$ = { a∈A : ‖ x (λ - a) ‖ ≤ k ‖ (λ - a) x ‖ for all x∈A and all λ∈ $\sigma_A(a)$ }, and they show that Q(A) ⊆Q_{ρ}(A). In [2] As'ad defined the extended center of a group G by:

Z_e(G) = { g∈G : gx = xg, for all x∈G except for a finite number}.

2. Extended Centerality

Definition 2.1. Let A be a unital complex Banach algebra.

1. The extended center of A is Z_e(A) = { a∈A : ax = xa, for all x∈A except for a finite number}.
2. The extended quasi center of A is Q_e(A) = $\underset{k\geq 1}{\cup}$ Q_e(k, A), where Q_e(k, A) = { a∈A : ‖ x (λ - a) ‖ ≤ k ‖ (λ - a) x ‖ for all x∈A except for a finite number and for all λ∈¢ }.
3. The extended σ-quasi center of A is $Q_{\sigma e}(\text{A}) = \underset{k\geq 1}{\cup} Q_{\sigma e}(\text{k}, \text{A})$, where $Q_{\sigma e}(\text{k}, \text{A})$ ={ a∈A: ‖ x (λ - a) ‖ ≤ k ‖ (λ - a) x ‖ for all x∈A except for a finite number and for all λ∈ $\rho_A(a)$ }.
4. The extended ρ-quasi center of A is $Q_{\rho e}(\text{A}) = \underset{k\geq 1}{\cup} Q_{\rho e}(\text{k}, \text{A})$, where $Q_{\rho e}(\text{k}, \text{A})$ = { a∈A: ‖ x (λ - a) ‖ ≤ k ‖ (λ - a) x ‖ for all x∈A except for a finite number and for all λ∈ $\sigma_A(a)$ }.

We start by the following proposition that is an elementary consequence of the definitions.

Proposition 2.2. *If A is a unital complex Banach algebra then,*

(i) Z_e(A) ⊆ Q_e(A) ⊆$Q_{\sigma e}$(A), *and* Q_e(A) ⊆$Q_{\rho e}$(A).

(ii) Q_e(A) = $Q_{\sigma e}(\text{A}) \cap Q_{\rho e}(\text{A})$.

(iii) *$Z_e(A)$ is a unital normed subalgebra of A,* $Z(A) \subseteq Z_e(A)$, $Q(A) \subseteq Q_e(A)$, $Q_\sigma(A) \subseteq Q_{\sigma e}(A)$*, and* $Q_\rho(A) \subseteq Q_{\rho e}(A)$.

Proof.

We prove the first part of (iii) (which is similar to the proof of Theorem 2.2(i) in [2]) and left the others to the reader.

Since A has a unity, say e then $e \in Z_e(A)$ and hence $Z_e(A) \neq \varphi$. Now, let a, $b \in Z_e(A)$ and α be a scalar. Then there are elements $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_m$ in A such that $xa = ax$ for all $x \in A \setminus \{ a_1, a_2, \ldots, a_n \}$ and $xb = bx$ for all $x \in A \setminus \{ b_1, b_2, \ldots, b_m \}$. Hence $(a\alpha + b)x = x(a\alpha + b)$ and $(ab)x = x(ab)$ for all $x \in A \setminus \{ a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \}$. Hence $(a\alpha + b)$ and ab belong to $Z_e(A)$. Therefore, $Z_e(A)$ is a a unital normed subalgebra of A□

Proposition 2.3. Let a be an element of a complex Banach algebra A, then $a \in Q_e(A)$ if and only if there is a constant L such that $\| ax - xa \| \leq L \| (\lambda - a)x \|$ for all $x \in A$ except a finite number of elements and for all $\lambda \in ¢$.

Proof.

Let $a \in Q_e(A)$, then there is $k \geq 1$ such that, for all $x \in A$ except a finite number of elements and for all $\lambda \in ¢$ we have, $\| x(\lambda - a) \| \leq k \| (\lambda - a)x \|$ from which we have, $\| ax - xa \| = \| x(\lambda - a) - (\lambda - a)x \| \leq \| x(\lambda - a) \| + \| (\lambda - a)x \| \leq k \| (\lambda - a)x \| + \| (\lambda - a)x \| = L \| (\lambda - a)x \|$ where, $L = k + 1$.

Conversely, suppose that there is a constant L such that $\| ax - xa \| \leq L \| (\lambda - a)x \|$ for all $x \in A$ except a finite number of elements and for all $\lambda \in ¢$. Then $\| x(\lambda - a) \| = \| (\lambda - a)x + ax - xa \| \leq \| (\lambda - a) x \| + \| ax - xa \| \leq \| (\lambda - a) x \| + L \| (\lambda - a)x \| = k \| (\lambda - a)x \|$, where $k = L + 1$.

Hence $a \in Q_e(A)$ □

Similar results hold for $Q_{\sigma e}(A)$ and $Q_{\rho e}(A)$, these results are given in the following two propositions, where their proofs are similar to the proof of proposition 2.3.

Proposition 2.4. Let a be an element of a complex Banach algebra A, then $a \in Q_{\sigma e}(A)$ if and only if there is a constant L such that $\| ax - xa \| \leq L \| (\lambda - a)x \|$ for all $x \in A$ except a finite number of elements and for all $\lambda \in \rho_A(a)$.

Proposition 2.5. Let a be an element of a complex Banach algebra A, then $a \in Q_{\rho e}$ (A) if and only if there is a constant L such that || ax – xa || ≤ L || (λ – a)x || for all x∈A except a finite number of elements and for all λ∈ $\sigma_A(a)$.

In the following theorem we show that $Q_e(1,A) = Z_e(A)$. This result and its proof is similar to the result Q(1,A) = Z(A) of C. Le Page in [6].

Theorem 2.6. Let A be a Banach algebra with unity over a complex field C and a ∈ A such that || x(λ − a) || ≤ || (λ − a)x || for all x ∈ A except a finite number of elements and for all λ ∈¢. Then a ∈ Z_e (A).

Proof. Choose | λ | > || a ||. Since A is a Banach algebra, then $|\lambda|^n > \|a\|^n \geq \|a^n\|$.

Hence the spectral radius $r(\frac{a}{\lambda}) = \lim_{n\to\infty} \|(\frac{a}{\lambda})^n\|^{\frac{1}{n}} < 1$, and so $(\lambda - a)^{-1}$ exists and $(\lambda - a)^{-1}y \in A$ for all y ∈ A. By this and the assumption (put $x = (\lambda - a)^{-1}y$) we get that for all y ∈ A except a finite number of elements we have

$$\|(\lambda - a)^{-1} y(\lambda - a)\| \leq \|(\lambda - a)(\lambda - a)^{-1}y\| = \|y\| \qquad (1).$$

Now for any fixed nonzero u ∈¢ and any positive integer n with $\frac{n}{|\mu|} > \|a\|$, by putting $\lambda = \frac{n}{\mu}$ in (1) we get $\|(\frac{n}{\mu} - a)^{-1} y(\frac{n}{\mu} - a)\| \leq \|y\|$ which implies that

$$\|(e - \frac{\mu}{n} a)^{-1} y(e - \frac{\mu}{n} a)\| \leq \|y\|.$$

By induction one can show that $\|(e - \frac{\mu}{n} a)^{-m} y(e - \frac{\mu}{n} a)^{m}\| \leq \|y\|$ for all m∈N,

so that $\|(e - \frac{\mu}{n} a)^{-n} y(e - \frac{\mu}{n} a)^{n}\| \leq \|y\|$.

Take the limit as n→ ∞ and use the continuity of the norm to get || exp(μa) y exp(-μa) || ≤ || y ||, but this inequility is also true for μ = 0, then one can easly see that the function f: ¢ → A defined by f(μ) = exp(μa) y exp(-μa) is bounded and entire. Hence by Liuoville's Theorem f is constant. Then f(μ) = exp(μa)y exp(−μa) = y.

Since y∈A was arbitrary in A except a finite number of elements, then, exp(μa)y = y exp(μa) for all such y and so $\sum_{0}^{\infty}\frac{(\mu a)^n}{n!}y = y\sum_{0}^{\infty}\frac{(\mu a)^n}{n!}$. This implies that μay = yμa for all μ∈¢, hence ay = ya for all y ∈ A except a finite number of elements. Therefore, a ∈ Z_e(A). □

Proposition 2.7. Let A be a complex Banach algebra with unity, k ≥ 1 and A^{-1} be the set of all invertible elements in A. If a∈Q_e(k, A)∩A^{-1}, then a^{-1}∈Q_e(k‖ a ‖ ‖ a^{-1} ‖, A).

Proof. Let a∈Q_e(k, A)∩A^{-1}. Then a^{-1}∈A, and for any μ∈¢\{0} and all x∈A except a finite number of elements we have, $\|x(\mu^{-1} - a^{-1})\| = \| x(\mu - a)\ (\mu\ a)^{-1}\| \leq \|x(\mu - a)\|\ |\ \mu^{-1}\ |\ \|\ a^{-1}\ \| \leq\ k\ \|\ (\mu - a)\ x\ \|\ |\ \mu^{-1}\ |\ \|\ a^{-1}\ \| = k\ \|\ (\mu\ a)\ (\mu^{-1} - a^{-1})x\ \|\ |\ \mu^{-1}\ |\ \|\ a^{-1}\ \| = k\ \|\ a(\mu^{-1} - a^{-1})x\ \|\ |\ \mu\ |\ |\ \mu^{-1}\ |\ \|\ a^{-1}\ \| \leq\ k\ \|\ a\ \|\ \|\ a^{-1}\ \|\ \|\ (\mu^{-1} - a^{-1})x\ \|$. However, μ∈¢\{0} iff μ^{-1}∈¢\{0}, then for any λ ∈¢\{0} and all x∈A except a finite number of elements we have, $\|\ x(\lambda - a^{-1})\ \| \leq k\ \|\ a\ \|\ \|\ a^{-1}\ \|\ \|\ (\lambda - a^{-1})x\ \|$.
Again a∈Q_e(k, A) implies that $\|\ xa^{-1}\| \leq \|\ x\ \|\ \|\ a^{-1}\| \leq \|\ a\ \|\ \|\ a^{-1}x\ \|\ \|\ a^{-1}\| \leq k\ \|\ a\ \|\ \|\ a^{-1}x\ \|\ \|\ a^{-1}\|$ for all x∈A except a finite number of elements. Together, $\|\ x(\lambda - a^{-1})\ \| \leq k\ \|\ a\ \|\ \|\ a^{-1}\ \|\ \|\ (\lambda - a^{-1})x\ \|$ for all λ ∈¢ and all x∈A except a finite number of elements. Therefore, a^{-1}∈Q_e(k‖ a ‖ ‖ a^{-1} ‖, A). □

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Received: December 11, 2017; Published: February 12, 2018