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Generalized Cone Metric Spaces and Fixed Point Theorems of Contractive Mappings

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Abstract

In this paper we shall acquaint with generalized cone metric spaces, contractive mappings will be considered and prove some classical fixed point theorems with the help of generalized cone metric space.

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1 Introduction

A classical and most cited theorem in theory of fixed point is Banach fixed point theorem [2] which is, if $(\Im, \Omega)$ is a complete metric space and a map $\Lambda : \Im \to \Im$ is contractive such as $\forall\ u, v \in \Im$

$$\Omega\left(\Lambda u, \Lambda v\right) \leq \Upsilon\ \Omega\left(u, v\right), \tag{1}$$

whenever $0 \leq \Upsilon < 1$, then Λ has a unique fixed point.
In [5] Kannan has proved a result which is, let $(\Im, \Omega)$ be a complete metric space and let $\Lambda : \Im \to \Im$ be a contractive map such as for all $u, v \in \Im$

$$\Omega\left(\Lambda u, \Lambda v\right) \leq \Upsilon \ \left\{\Omega(u, \Lambda u) + \Omega(v, \Lambda v)\right\}, \tag{2}$$

where $0 \leq \Upsilon < \frac{1}{2}$, then Λ has a unique fixed point.
After it, in [6] Chatterjea has proved a theorem which is, let $(\Im, \Omega)$ be a complete metric space and let $\Lambda : \Im \to \Im$ be a contractive map such as for all $u, v \in \Im$

$$\Omega\left(\Lambda u, \Lambda v\right) \leq \Upsilon \ \left\{\Omega(u, \Lambda v) + \Omega(v, \Lambda u)\right\}, \tag{3}$$

where $0 \leq \Upsilon < \frac{1}{2}$, then Λ has a unique fixed point.
latter on in [7] Rezapour has proved a result which is, let $(\Im, \Omega)$ be a complete cone metric space and a contractive mapping $\Lambda : \Im \to \Im$ such as

$$\Omega\left(\Lambda u, \Lambda v\right) \leq \Upsilon \ \Omega\left(u, v\right) + \beta \ \Omega\left(v, \Lambda u\right), \tag{4}$$

for all $u, v \in \Im$, where $0 \leq \Upsilon, \beta < 1$ are constants moreover $\beta + \Upsilon < 1$, then Λ has a unique fixed point.
Now define the cone metric space. Suppose a Banach space B, and $C \subset B$ is called cone if
(i) $\emptyset \neq C \neq \{0\}$ and closed.
(ii) $uu_1 + vu_2 \in C$ for all $u_1, u_2 \in C$ and $0 \leq u, v \in \mathbf{R}$.
(iii) The intersection of C and $-C$ is always equal to $\{0\}$.
Given a cone $C \subset B$, we can consider a partial ordering $\leq'$ on C such as $u \leq v$ iff $v - u \in C$, while $u \ll v$ iff $v - u \in C^\circ$ where C° stands for interior of C.
The cone C is called normal cone if $\exists\, \tau > 0$ such as for all $u, v \in B$, $0 \leq u \leq v$ implies that $\|u\| \leq \tau \|v\|$, and the least positive number τ which satisfies this condition is called the normal constant of cone C.
The cone C is said to be regular cone, if every increasing sequence in cone C which is bounded from above is convergent, similarly if every decreasing sequence which is bounded from below is convergent. The sequence $\{u_{\eta \geq 1}\} \in C$ is convergent sequence if $u_1 \leq u_2 \leq ... \leq v$ for some $v \in B$ then there is $u \in B$ such that $\|u_\eta - u\| \to 0$ whenever $\eta \to \infty$. It has been proved in [7] every regular cone is normal cone.
In this paper we always consider B be a Banach space, $C \subset B$ be a cone with $C^o \neq \emptyset$ and partial ordering $\leq'$ with respect to C which is define above.
To understand the standard notations and definitions on cone metric space reader must review [3, 7].
Let $\Im \neq \emptyset$ be a set. A mapping $\Omega : \Im \times \Im \to \mathbf{R}$ satisfies
(i) $\Omega\left(u_1, u_2\right) > 0$ for all $u_1, u_2 \in \Im$ and $\Omega\left(u_1, u_2\right) = 0$ if and only if $u_1 = u_2$.
(ii) $\Omega\left(u_1, u_2\right) = \Omega\left(u_2, u_1\right)$ for all $u_1, u_2 \in \Im$.
(iii) $\Omega\left(u_1, u_2\right) \leq \Omega\left(u_1, u_3\right) + \Omega\left(u_3, u_4\right) + \Omega\left(u_4, u_2\right)$ (rectangular inequality [1, 4])

for all $u_1, u_2 \in \Im$ and $u_3, u_4 \in \Im - \{u_1, u_2\}$.
Then Ω is called a generalized cone metric on $\Im$, and $(\Im, \Omega)$ is called a generalized cone metric space.
Let $(\Im, \Omega)$ be a generalized cone metric space and it is complete then we shell call it complete generalized cone metric space. Let see an example of a space which is generalized cone metric space but not standard cone metric space.
Example 1.1
Let $\Im = B = \mathbf{R}$, $\Upsilon \in (0, \infty)$ and $C = [0, \infty)$. Define $\Omega : \Im \times \Im \to \mathbf{R}$ as follows

$$\Omega(u,v) = \begin{cases} 0, & \text{if} & u = v, \\ 3\Upsilon, & \text{if u and v are in \{2,3\}} & u \neq v, \\ \Upsilon, & \text{if u and v not both at a time in \{2,3\}} & u \neq v. \end{cases}$$

Its easy for the reader to check that $(\Im, \Omega)$ is a generalized cone metric space but is not standard cone metric space, it not full fill the triangular property $\Omega(3,2) = 3\Upsilon$ and $\Omega(3,1) + \Omega(1,2) = 2\Upsilon$, for the triangular inequality we must have to prove that $3\Upsilon \leq 2\Upsilon$ but according to our partial ordering $2\Upsilon - 3\Upsilon = -\Upsilon$ must present in C, which cannot be happen.

2 Main results

In this section we shell prove our main results.
Theorem 2.1
Let $(\Im, \Omega)$ be complete generalized cone metric space, C is the normal cone and τ be the normal constant. A mapping $\Lambda : \Im \to \Im$ satisfies the condition 1, then Λ has a unique fixed point.
Proof:
Let $u_0 \in \Im$, if $u_0 = \Lambda u_0$ then we have done, if $u_0 \neq \Lambda u_0$ then we can define a sequence such as $\Lambda u_0 = u_1$, $\Lambda u_1 = u_2 = \Lambda^2 u_0$,...,$\Lambda u_{\eta-1} = u_\eta = \Lambda^\eta u_0$,....We have

$$\begin{aligned} \Omega(u_{\eta+1}, u_\eta) &= \Omega(\Lambda u_\eta, \Lambda u_{\eta-1}) \leq \Upsilon\, \Omega(u_\eta, u_{\eta-1}), \\ &\leq \Upsilon^2\, \Omega(u_{\eta-1}, u_{\eta-2}), \\ &. \\ &. \\ &. \\ &\leq \Upsilon^\eta\, \Omega(u_1, u_0). \end{aligned}$$

Now for $\eta > \rho$,

$$\Omega(u_\eta, u_\rho) \leq \Omega(u_\eta, u_{\eta-1}) + \Omega(u_{\eta-1}, u_{\eta-2}) + ... + \Omega(u_{\rho+1}, u_\rho),$$

$$\begin{aligned} &\leq \left(\Upsilon^{\eta-1} + \Upsilon^{\eta-2} + \dots + \Upsilon^{\rho}\right) \Omega\left(u_1, u_0\right), \\ &\leq \frac{\Upsilon^{\rho}}{1-\Upsilon}\, \Omega\left(u_1, u_0\right). \end{aligned}$$

Which means that $\Omega\left(u_\eta, u_\rho\right) \to 0$ whenever $\eta, \rho \to \infty$, this implies that our sequence $\{u_\eta\}$ is cauchy sequence. As $\Im$ is complete so $\exists\ u^{'} \in \Im$ such that $\|u_\eta - u^{'}\| \to 0$ whenever $\eta \to \infty$. Since

$$\begin{aligned} \Omega(\Lambda u^{'}, u^{'}) &\leq \Omega(\Lambda u^{'}, \Lambda u_\eta) + \Omega(\Lambda u_\eta, \Lambda u_{\eta-1}) + \Omega(\Lambda u_{\eta-1}, u^{'}), \\ &\leq \Upsilon\ \Omega(u^{'}, u_\eta) + \Upsilon^{\eta}\ \Omega(u_1, u_0) + \Omega(u_\eta, u^{'}). \end{aligned}$$

So, we have

$$\|\Omega(\Lambda u^{'}, u^{'})\| \leq \Upsilon\ \tau\ \|\Omega(u^{'}, u_\eta)\| + \Upsilon^{\eta}\ \tau\ \|\Omega(u_1, u_0)\| + \tau\ \|\Omega(u_\eta, u^{'})\|.$$

As $\Upsilon\ \tau\ \|\Omega(u^{'}, u_\eta)\| + \Upsilon^{\eta}\ \tau\ \|\Omega(u_1, u_0)\| + \tau\ \|\Omega(u_\eta, u^{'})\| \to 0$ whenever $\eta \to \infty$. So $\|\Omega(\Lambda u^{'}, u^{'})\| = 0$, this means that $u^{'}$ is a fixed point of Λ. Now let $u^{''} \in \Im$ be another fixed point of Λ, then

$$\Omega(u^{'}, u^{''}) = \Omega(\Lambda u^{'}, \Lambda u^{''}) \leq \Upsilon\ \Omega(u^{'}, u^{''}).$$

This means that $\|\Omega(u^{'}, u^{''})\| = 0$, so $u^{'} = u^{''}$.

Theorem 2.2

Let $(\Im, \Omega)$ be a complete generalized cone metric space, C be a normal cone having normal constant τ. For $c^{'} \in B$ with $c^{'} \in C^o$ and $u_0 \in \Im$, define a ball $B(u_0, c^{'}) = \{u \in \Im \mid \Omega(u_0, u) \leq c^{'}\}$. Suppose $\Lambda : \Im \to \Im$ be a mapping such as it holds condition 1 for all $u, v \in B(u_0, c^{'})$ and $\Omega(\Lambda u_0, u_0) \leq (\frac{1-\Upsilon}{2})c^{'}$, also $\Omega(\Lambda u_0, \Lambda u_1) \leq (\frac{1-\Upsilon}{2})c^{'}$, then Λ has a unique fixed point in $B(u_0, c^{'})$.

Proof:

First of all we have to prove that $B(u_0, c^{'})$ is complete. Let $\{u_\eta\}$ be a cauchy sequence in $B(u_0, c^{'})$. So $\{u_\eta\}$ is also cauchy sequence in $\Im$. As $\Im$ is complete so $\exists\ u^{'} \in \Im$ such that $\|u_\eta - u^{'}\| \to 0$ when $\eta \to \infty$. Now

$$\Omega(u^{'}, u_0) \leq \Omega(u^{'}, u_\eta) + \Omega(u_\eta, u_{\eta-1}) + c^{'}.$$

Since $\Omega(u^{'}, u_\eta) + \Omega(u_\eta, u_{\eta-1}) \to 0$ when $\eta \to \infty$, so

$$\Omega(u_\eta, u^{'}) \leq c^{'}.$$

This implies that $B(u_0, c^{'})$ is complete. Now we have only need to show that $\Lambda u \in B(u_0, c^{'})$, whenever $u \in B(u_0, c^{'})$.

$$\begin{aligned} \Omega(u_0, \Lambda u) &\leq \Omega(u_0, \Lambda u_0) + \Omega(\Lambda u_0, \Lambda u_1) + \Omega(\Lambda u_1, \Lambda u), \\ &\leq \left(\frac{1-\Upsilon}{2}\right) c^{'} + \left(\frac{1-\Upsilon}{2}\right) c^{'} + c^{'}\ \Upsilon = c^{'}, \end{aligned}$$

which completes our proof.
Theorem 2.3
Let $(\Im, \Omega)$ be complete generalized cone metric space, C is the normal cone and τ be the normal constant. A mapping $\Lambda : \Im \to \Im$ satisfies the condition 2, then Λ has a unique fixed point.
Proof:
Let $u_0 \in \Im$, if $u_0 = \Lambda u_0$ then we have done, if $u_0 \neq \Lambda u_0$ then we can define a sequence such as $\Lambda u_0 = u_1$, $\Lambda u_1 = u_2 = \Lambda^2 u_0$,...,$\Lambda u_{\eta-1} = u_\eta = \Lambda^\eta u_0$,....We have

$$\begin{aligned}
\Omega(u_{\eta+1}, u_\eta) &= \Omega\left(\Lambda u_\eta, \Lambda u_{\eta-1}\right) \leq \Upsilon\{\Omega(\Lambda u_\eta, u_\eta) + \Omega(\Lambda u_{\eta-1}, u_{\eta-1})\}, \\
&\leq \frac{\Upsilon}{1-\Upsilon}\ \Omega(u_\eta, u_{\eta-1}), \\
&. \\
&. \\
&. \\
&\leq \Gamma^\eta\ \Omega(u_1, u_0).
\end{aligned}$$

where $\Gamma = \frac{\Upsilon}{1-\Upsilon}$ and clearly $\Gamma < 1$.
For $\eta > \rho$,

$$\begin{aligned}
\Omega\left(u_\eta, u_\rho\right) &\leq \Omega\left(u_\eta, u_{\eta-1}\right) + \Omega\left(u_{\eta-1}, u_{\eta-2}\right) + ... + \Omega\left(u_{\rho+1}, u_\rho\right), \\
&\leq \left(\Gamma^{\eta-1} + \Gamma^{\eta-2} + ... + \Gamma^\rho\right)\Omega\left(u_1, u_0\right), \\
&\leq \frac{\Gamma^\rho}{1-\Gamma}\ \Omega\left(u_1, u_0\right).
\end{aligned}$$

So, we get

$$\|\Omega\left(u_\eta, u_\rho\right)\| \leq \frac{\Gamma^\rho}{1-\Gamma}\ \tau\|\Omega\left(u_1, u_0\right)\|.$$

Which means that $\Omega\left(u_\eta, u_\rho\right) \to 0$ whenever $\eta, \rho \to \infty$ which implies that our sequence $\{u_\eta\}$ is cauchy sequence. As $\Im$ is complete, so $\exists\ u' \in \Im$ such that $\|u_\eta - u'\| \to 0$ whenever $\eta \to \infty$. Since

$$\begin{aligned}
\Omega(\Lambda u', u') &\leq \Omega(\Lambda u', \Lambda u_\eta) + \Omega(\Lambda u_\eta, \Lambda u_{\eta-1}) + \Omega(\Lambda u_{\eta-1}, u'), \\
&\leq \Gamma^{\eta+1}\ \Omega(u_{u_0}, u_1) + \frac{\Gamma^\eta}{1-\Upsilon}\ \Omega(u_1, u_0) + \frac{1}{1-\Upsilon}\ \Omega(u_\eta, u').
\end{aligned}$$

So, we have

$$\|\Omega(\Lambda u', u')\| \leq \Gamma^{\eta+1}\ \tau\ \|\Omega(u_0, u_1)\| + \frac{\Gamma^\eta}{1-\Upsilon}\ \tau\ \|\Omega(u_1, u_0)\| + \frac{1}{1-\Upsilon}\ \tau\ \|\Omega(u_\eta, u')\|.$$

As
$\Gamma^{\eta+1}\ \tau\ \|\Omega(u_0,u_1)\| + \frac{\Gamma^{\eta}}{1-\Upsilon}\ \tau\ \|\Omega(u_1,u_0)\| + \frac{1}{1-\Upsilon}\ \tau\ \|\Omega(u_\eta,u^{'})\| \to 0$ when $\eta \to \infty$.
So $\|\Omega(\Lambda u^{'},u^{'})\| = 0$ this means that $u^{'}$ is a fixed point of Λ. Now let $u^{''} \in \Im$ be another fixed point of Λ, then

$$\Omega(u^{'},u^{''}) = \Omega(\Lambda u^{'},\Lambda u^{''}) \leq \Upsilon\left(\Omega(\Lambda u^{'},u^{'}) + \Omega(\Lambda u^{''},u^{''})\right) = 0.$$

Hence $\|\Omega(u^{'},u^{''})\| = 0$ which means that $u^{'} = u^{''}$.

Theorem 2.4

Let $(\Im,\Omega)$ be complete generalized cone metric space, C is the normal cone and τ be the normal constant. A mapping $\Lambda : \Im \to \Im$ satisfies the condition 3, then Λ has a unique fixed point.

Proof:

Let $u_0 \in \Im$, if $u_0 = \Lambda u_0$ then we have done, if $u_0 \neq \Lambda u_0$ then we can define a sequence such as $\Lambda u_0 = u_1$, $\Lambda u_1 = u_2 = \Lambda^2 u_0$,...,$\Lambda u_{\eta-1} = u_\eta = \Lambda^\eta u_0$,....We have

$$\begin{aligned}
\Omega(u_{\eta+1},u_\eta) &= \Omega\left(\Lambda u_\eta,\Lambda u_{n-1}\right) \leq \Upsilon\{\Omega(\Lambda u_\eta,u_{\eta-1}) + \Omega(u_\eta,\Lambda u_{\eta-1})\},\\
&\leq \Upsilon\ \{\Omega(\Lambda u_\eta,u_\eta) + \Omega(u_\eta,u_{\eta-1})\},\\
&\leq \frac{\Upsilon}{1-\Upsilon}\ \Omega(u_\eta,u_{\eta-1}),\\
&\ .\\
&\ .\\
&\ .\\
&\leq \Gamma^{\eta}\ \Omega(u_1,u_0).
\end{aligned}$$

Where $\Gamma = \frac{\Upsilon}{1-\Upsilon}$ and clearly $\Gamma < 1$. Its easy to show that this sequence $\{u_\eta\}$ is a cauchy sequence, so $\exists\ u^{'} \in \Im$ such as $\|u_\eta - u^{'}\| \to 0$. Since

$$\begin{aligned}
\Omega(\Lambda u^{'},u^{'}) &\leq \Omega(\Lambda u^{'},\Lambda u_\eta) + \Omega(\Lambda u_\eta,\Lambda u_{\eta-1}) + \Omega(\Lambda u_{\eta-1},u^{'}),\\
&\leq \Upsilon\left(\Omega(\Lambda u^{'},u_\eta) + \Omega(u^{'},\Lambda u_\eta)\right) + \Gamma^{\eta}\Omega(u_1,u_0) + \Omega(u_\eta,u^{'}),\\
&\leq \Upsilon\left(\Omega(\Lambda u^{'},u^{'}) + \Omega(u^{'},u_\eta) + \Omega(u^{'},\Lambda u_\eta)\right) + \Gamma^{\eta}\Omega(u_1,u_0) + \Omega(u_\eta,u^{'}),\\
\Omega(\Lambda u^{'},u^{'}) &\leq \frac{\Upsilon}{1-\Upsilon}\Omega(u_\eta,u^{'}) + \frac{\Upsilon}{1-\Upsilon}\Omega(u_{\eta+1},u^{'}) + \frac{\Gamma^{\eta}}{1-\Upsilon}\Omega(u_1,u_0) + \frac{1}{1-\Upsilon}\Omega(u_\eta,u^{'}).
\end{aligned}$$

So, we have

$$\begin{aligned}
\|\Omega(\Lambda u^{'},u^{'})\| \leq \frac{\Upsilon}{1-\Upsilon}\ \tau\|\Omega(u_\eta,u^{'})\| + \frac{\Upsilon}{1-\Upsilon}\ \tau\|\Omega(u_{\eta+1},u^{'})\| &+ \frac{\Gamma^{\eta}}{1-\Upsilon}\ \tau\|\Omega(u_1,u_0)\|\\
&+ \frac{1}{1-\Upsilon}\ \tau\|\Omega(u_\eta,u^{'})\|.
\end{aligned}$$

Clearly $\|\Omega(\Lambda u^{'}, u^{'})\| = 0$, so $\Lambda u^{'} = u^{'}$ is fixed point of Λ. Now let $u^{''} \in \Im$ be another fixed point of Λ , then

$$\Omega(u^{'}, u^{''}) = \Omega(\Lambda u^{'}, \Lambda u^{''}) \leq \Upsilon \left(\Omega(\Lambda u^{'}, u^{''}) + \Omega(u^{'}, \Lambda u^{''})\right) = 2\Upsilon\ \Omega(u^{'}, u^{''}).$$

Hence $\|\Omega(u^{'}, u^{''})\| = 0$, so $u^{'} = u^{''}$.

Theorem 2.5

Let $(\Im, \Omega)$ be complete generalized cone metric space, C is the normal cone and τ be the normal constant. A mapping $\Lambda : \Im \to \Im$ satisfies the condition 4, then Λ has a unique fixed point.

Proof:

Let $u_0 \in \Im$, if $u_0 = \Lambda u_0$ then we have done, if $u_0 \neq \Lambda u_0$ then we can define a sequence such as $\Lambda u_0 = u_1$, $\Lambda u_1 = u_2 = \Lambda^2 u_0$,...,$\Lambda u_{\eta-1} = u_\eta = \Lambda^\eta u_0$,....We have

$$\begin{aligned}
\Omega(u_{\eta+1}, u_\eta) &= \Omega\left(\Lambda u_\eta, \Lambda u_{\eta-1}\right) \leq \Upsilon\ \Omega\left(u_\eta, u_{\eta-1}\right) + \beta\ \Omega(u_\eta, \Lambda u_{\eta-1}),\\
&\leq \Upsilon\Omega(u_\eta, u_{\eta-1}),\\
&\leq \Upsilon^2\ \Omega\left(u_{\eta-1}, u_{\eta-2}\right),\\
&.\\
&.\\
&.\\
&\leq \Upsilon^\eta\ \Omega\left(u_1, u_0\right).
\end{aligned}$$

Its easy to check that this sequence $\{u_\eta\}$ is a cauchy sequence. As $\Im$ is complete so $\exists\ u^{'} \in \Im$ such that $\|u_\eta - u^{'}\| \to 0$ whenever $\eta \to \infty$. Since

$$\begin{aligned}
\Omega(\Lambda u^{'}, u^{'}) &\leq \Omega(\Lambda u^{'}, \Lambda u_\eta) + \Omega(\Lambda u_\eta, \Lambda u_{\eta-1}) + \Omega(\Lambda u_{\eta-1}, u^{'}),\\
&\leq \Upsilon\ \Omega(u^{'}, u_\eta) + \beta\ \Omega(u^{'}, \Lambda u_\eta) + \Upsilon^\eta \Omega(u_0, u_1) + \Omega(\Lambda u_\eta, u^{'}).
\end{aligned}$$

So, we have

$$\|\Omega(\Lambda u^{'}, u^{'})\| \leq \Upsilon\ \tau\ \|\Omega(u^{'}, u_\eta)\| + \beta\ \tau\ \|\Omega(u^{'}, \Lambda u_\eta)\| + \Upsilon^\eta\ \tau\ \|\Omega(u_0, u_1)\| + \tau\ \|\Omega(\Lambda u_\eta, u^{'})\|.$$

As $\Upsilon\ \tau\ \|\Omega(u^{'}, u_\eta)\| + \beta\ \tau\ \|\Omega(u^{'}, \Lambda u_\eta)\| + \Upsilon^\eta\ \tau\ \|\Omega(u_0, u_1)\| + \tau\ \|\Omega(\Lambda u_\eta, u^{'})\| \to 0$ whenever $\eta \to \infty$. So, $\|\Omega(\Lambda u^{'}, u^{'})\| = 0$, this means that $u^{'}$ is a fixed point of Λ. Now let $u^{''} \in \Im$ be another fixed point of Λ, then

$$\begin{aligned}
\Omega(u^{'}, u^{''}) &= \Omega(\Lambda u^{'}, \Lambda u^{''}) \leq \Upsilon\ \Omega(u^{'}, u^{''}) + \beta\ \Omega(u^{'}, \Lambda u^{''}).\\
&= (\Upsilon + \beta)\ \ \Omega(u^{'}, u^{''}) < \Omega(u^{'}, u^{''}).
\end{aligned}$$

This implies that $\|\Omega(u^{'}, u^{''})\| = 0$, so $u^{'} = u^{''}$, so the fixed point is unique.

References

[1] A. Akbar and M. Arshad, Kannan fixed point theorem on generalized metric spaces, *The Journal of Nonlinear Sciences and Its Applications*, **1** (2008), no. 1, 45-48.

[2] Kazimierz Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Vol. 28, Cambridge University Press, 1990. https://doi.org/10.1017/cbo9780511526152

[3] H. L. Guang, and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *Journal of Mathematical Analysis and Applications*, **332** (2007), no. 2, 1468-1476. https://doi.org/10.1016/j.jmaa.2005.03.087

[4] M. Balaiah, A fixed point theorem in generalized metric spaces, *International Journal of Scientific and Innovative Mathematical Research (IJSIMR)*, **3** (2015), no. 12, 24-26.

[5] R. Kannan, Some results on fixed points, *Bull. Cal. Math. Soc.*, **60** (1968), 71-76.

[6] S. K. Chatterjea, Fixed-point theorems, *Dokladi na Bolgarskata Akademiya na Naukite*, **25** (1972), no. 6.

[7] S. Rezapour and R. Hamlbarani, Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", *Journal of Mathematical Analysis and Applications*, **345** (2008), no. 2, 719-724. https://doi.org/10.1016/j.jmaa.2008.04.049

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