International Mathematical Forum, Vol. 13, 2018, no. 2, 91 - 97
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2018.712100

Strongly Rickart $*$-Rings

Tamadher Arif Ibrahiem

Department of Mathematics
College of Science for women
University of Baghdad, Iraq

Abstract

A $*$ - ring A is said to be strongly Rickart $*$- ring if $x^r = pA$ for left semi-central projection $p \in A$ and for all $x \in A$. In this paper, the concept of Strongly Rickart $*$-ring has been studied with its properties. Let A be $*$-ring , then every strongly Rickart $*$-ring is proving p. q. -Baer $*$-ring. The converse is true if $C(ay) = C(a)C(y)$ for all $a, y \in A$. For a $*$- ring A : A is a biregular and strongly Rickart $*$-ring if and only if A is a $*$-biregular with proper involution $*$ if and only if for all $y \in A$. The principal ideal of yA is generated by central projection p such that $yA = pA$. Some characterization of strongly Rickart $*$-ring and some examples are given.

Mathematics Subject Classification: 16W10

Keywords: Strongly Rickart $*$-rings; strongly Rickart rings; Rickart $*$-rings, *-biregular $*$-rings

1 Introduction

Throughout this paper, all rings are associative with unity. $P(A)$ is the set of all projection elements of A and $a^r = \{x \in A | ax = 0\}$ (resp. $a^\ell = \{x \in A | xa = 0\}$) is the right (resp. left) annihilator of $x \in A$. An idempotent element p is called right (resp. left) semi-central if $pa = pap$ (resp. $ap = pap$), for all $a \in A$ [5]. $p \in A$ is called central if $pa = ap$ each $a \in A$ [5]. Berberian defined in [9], a ring A is a ring with involution (or $*$-ring) if there exist an operation $* : A \longrightarrow A$: $(a^*)^* = a$, $(a + b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. An idempotent (hence a right (left) semi-central idempotent) element p ($p^2 = p$) is projection if it's self adjoint ($p^* = p$) [9]. According to ref. [11] a ring A have IFP (resp. $*$-IFP) if for every $a \in A$ and a^r is an ideal (resp. $*$-ideal) of A. Since, $a = (a^*)^r$ for $*$-rings have $*$-IFP, it follows that a is $*$-ideal. If a $*$-ring A having $*$-IFP, it has IFP but the converse is not true in general. The ring (resp. $*$-ring) A is an Abelian (resp.$*$-Abelian) if all its idempotent (resp. projection) elements are central and A is reduced if it has no nonzero nilpotent elements [11]. A ring (resp. $*$-ring) A is called symmetric (resp. $*$-symmetric) if for any elements $abc = 0$, then $acb = 0$ (resp. $acb^* = 0$) for all a,b and $c \in A$[12]. From ref. [12], every - symmetric ring with 1 is symmetric while the converse cannot be true in general, even for the commutative rings. It's well known that every symmetric rings are abelian [3]. A ring (resp. $*$-ring) A is biregular (resp. $*$-biregular) if every principal ideal (resp. $*$-ideal) of A is generated by a central idempotent (resp. projection) [11]. From this reference, a biregular $*$-ring A is $*$-biregular. A $*$-ring A is to be special almost $*$-clean if each element a has the form $a = u + e$ for a regular element u and a projection p with $aA \cap pA = 0$ [4]. A ring (resp. $*$-ring) A is to be Rickart [1] (resp. $*$-Rickart [10]) if the right annihilator of any single element in A is generated by idempotent (resp. projection) element. In a Rickart $*$-ring, the concepts: abelian, $*$-abelian, reduced, IFP and $*$-IFP are equivalent [11]. A ring A is to be strongly Rickart if the right annihilator of each single element in A is generated by left semi-central idempotent of A [8]. Clearly, every strongly Rickart ring is Rickart [8]. The aim of this paper is to define a strongly Rickart $*$-ring similarly in a strongly Rickart ring by replacing the idempotent element by a projection and proved that a $*$-ring A is strongly Rickart $*$-ring if and only if A is a Rickart $*$-ring has $*$-IFP(resp. A is a $*$-abelian). Finally, the author in [7] introduced the strong Rickart ring as a generalization to strongly Rickart Ring and to a Rickart $*$-ring.

2 Strongly Rickart $*$ - ring

Definition 2.1 *A *-ring A is to be strongly Rickart $*$-ring if $x^r = pA$ for left semi-central projection $p \in A$ and for all $x \in A$.*

3 Main Results

Proposition 3.1 *Let A be Strongly Rickart $*$-ring and $y \in A$. Then there exists a unique right semi-central projection p such that (i) $yp = y$ and (ii) $yz = 0$ iff $pz = 0$. Similarly, there exists a unique left semi-central projection q such that (iii) $qx = x$, and (iv) $yx = 0$ iff $yq = 0$.*
Proof. Similar to proof of ([9], Proposition 3, p.13).

Remarks 3.2 *1. According to ([6], Proposition 6(ii)) if $p = p^2$ and $p = p^*$ in a $*$-ring A, then p is left(right) semi-central if and only if p is central.*

2. *According to (1) a $*$-ring A is strongly Rickart $*$-ring if $x^r = pA$ for central element $p \in P(A)$ and for all $x \in A$.*

3. *Every strongly Rickart $*$-ring is a strongly Rickart ring (and hence gives all its generalizations). The converse is not true in genera (see Example 3.4)*

4. *Every strongly Rickart $*$-ring is a Rickart $*$-ring (and hence is Rickart ring). The converse is not true in general (see Example 3.6)*

Proposition 3.3 *([11], Proposition 10). Let A be a Rickart $*$-ring, then the following conditions are equivalent:*

1. *A is reduced.*
2. *A is $*$-abelian.*
3. *A is abelian.*
4. *A has $*$- IFP.*
5. *A has IFP.*

Proof. see ([11], proposition 10).

The previous proposition is not true in general, (without the condition Rickart $*$-ring). It is consider the following example which appear in ([11], Example 2).

Example 3.4 *Let $A = F \bigoplus F$ be a $*$-ring where F be a field, with involution $(x, y)^* = (y, x)$ for all$x, y \in F$. A has IFP (and hence it's abelian) but does not have $*$-IFP. It's obvious that A is Rickart (and hence is strongly Rickart) but not Rickart $*$- ring.*

Proposition 3.5 *Every strongly Rickart $*$-ring has $*$-IFP (resp. $*$-abelian). Proof. follows (Remarks 3.2(3)), ([8], Proposition 1.4), (Remarks 3.2(4)) and Proposition 3.3*

Example 3.6 *Let $A = M_2(Z_3)$. The involution $*$ on A defined by the transpose. By ([6], Example 2.4), A is a Rickart $*$- ring. The set of the idempotent elements in A is ([9], Exercises 17A, p.10)* $\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}\}$

where $bc = a - a^2$ *and* $P(A) = \{0, 1, e, f, 1-e, 1-f\}$ *where* $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $f = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. *Since 0 and 1 are the only central projections in A. Hence, A is not* $*$*-abelian, hasn't* $*$*-IFP (Proposition 3.3) and is not strongly Rickart* $*$*-ring. Recall that every idempotent of a* $*$*-ring having* $*$*-IFP are a central projection ([11], Proposition 8). Hence, the following Theorem hold.*

Theorem 3.7 *A* $*$*-ring A is strongly Rickart* $*$*-ring if and only if A is a Rickart* $*$*-ring has* $*$*-IFP(resp. A is a* $*$*-abelian).*

The properties of strongly Rickart $*$-ring in Theorem 3.7 give us more than one corollary.

Corollary 3.8 *If A is a strongly Rickart* $*$*-ring, then for every* $x \in A$*, there is a central projection* $p \in A$ *such that* $(xA)^r = (pA)^r$.
Proof. let A be a strongly Rickart $*$*-ring and* $x \in A$*. So there is a projection q such that* $x^r = qA$ *(clearly that q is central). From Theorem 3.7,* $(Ax)^r = x^r = (xA)^r = qA = ((1-q)A)^r$*. Put* $p = 1 - q$*, clearly that p is central projection.*

Corollary 3.9 *A* $*$*-ring A is strongly Rickart* $*$*-ring if and only if A is a Rickart* $*$*-ring has IFP(resp. A is an abelian).*

Corollary 3.9 shows that every idempotent of strongly Rickart $*$-ring is central.

S. K. Berberian in ([10], 3.8) prove that in a Rickart $*$-ring, every central idempotent is projection. Hence, the following Corollary holds.

Corollary 3.10 *Every idempotent of a strongly Rickart* $*$*-ring is central projection.*
Proof. follows Corollary 3.9 and ([10], 3.8).

Corollary 3.11 *Let A be a* $*$*-ring. The following statements are equivalent:*

1. *A is a strongly Rickart* $*$*-ring.*

2. *A is a strongly Rickart ring and* $Ae = Ae^*e$ *for every idempotent e*

Proof. $(1 \Rightarrow 2)$ *by remark 3.2(1), A is strongly Rickart ring. By remark 3.2(3) and ([10], Proposition 1.11) the second condition hold.*
$(2 \Rightarrow 1)$*from [8], A is Rickart ring and by ([10], Proposition 1.11), A is a Rickart* $*$*- ring. Since, A is an abelian ring and by Corollary 3.10 the proof is complete.*

Proposition 3.12 *A $*$-ring A is strongly Rickart $*$-ring if and only if A is $*$-abelian and special almost $*$-clean.*
Proof. $\Rightarrow$) *It's clear that from Theorem 3.7, A is $*$-abelian and Rickart $*$-ring. So, by ([4], Theorem 5.2), A is a special almost *-clean.*
$\Leftarrow$) *From ([4], Theorem 5.2), A is a Rickart $*$-ring. Hence, A is a strongly Rickart $*$-ring (Theorem 3.7).*

Recall that from [2], a $*$-ring A is to be a p.q.-Baer $*$-ring if, for every principal right ideal xA, $(xA)^r = pA$, where p is a projection in A. An element a in a $*$-ring possesses a central cover if there exists a smallest central projection p such that $pa = a$. If such projection p exists, then it is unique, and is called the central cover of a, denoted by $p = C(a)$ or $p = CR(a)$ [2].

Proposition 3.13 *Every strongly Rickart $*$-ring is p.q.-Baer $*$-ring. The converse is true if $C(ay) = C(a)C(y)$ for all $a, y \in A$.*
Proof. Suppose that A is strongly Rickart $*$-ring. Since, A has a unity element where A is Rickart $*$-ring [10], then by Corollary 3.9 and ([2], Proposition 3.1), the proof is complete. For the converse, A is a p.q.-Baer $*$-ring. For any elements a, y the $C(ay) = C(a)C(y)$, then A is Rickart $*$-ring ([2], Corollary 2.7). From the proof of ([2], Corollary 2.7), A has $*$-IFP. Hence, A is strongly Rickart $*$-ring (Corollary 3.9).

According to [12], the a $*$- ring A is - symmetric if whenever $abc = 0$, then $acb^* = 0$, for all elements $a, b, c \in A$. Furthermore, a $*$- ring A is symmetric if its $*$- symmetric ring with 1 but even for the commutative rings and the converse need not be true in general.

Example 3.14 *The ring $A = Z_p \oplus Z_p$ with component-wise addition and multiplication is symmetric ([12], Examples 2.1) and Rickart ring, for any prime integers p. So, A is a strongly Rickart ring. The exchange involution $* : x \to x^*$ on A define by $(x, y)^* = (y, x)$ for all $(x, y) \in Z_p \oplus Z_p$. Now, for nonzero elements $x, y, z, n \in Z_p (x, 0)(0, y)(z, n) = (0, 0)$ while $(x, 0)(z, n)(0, y)^* = (xz, 0)(y, 0)(0, 0)$. Therefore, $A = Z_p \oplus Z_p$ with the involution $*$ is not $*$-symmetric ([12], Examples 2.1).*

Lemma 3.15 *([9], Exercises 7C, p.9) If A is a symmetric $*$-ring, then for every idempotent $e \in A$ there exists a projection $f \in A$ such that $eA = fA$.*

Proposition 3.16 *Let A be a symmetric $*$- ring. If A is a Rickart $*$- ring, then A is a strongly Rickart $*$-ring.*
Proof. by ([12], Lemma 1.2(3)), A is a symmetric ring and by [3], A is an abelian ring. By Proposition 3.3, A is an abelian $*$- ring. Hence A is a strongly Rickart $*$- ring (Theorem 3.7).

Recall that a ring A, with center Z, is compressible if, for every idempotent $e \in A$, the center of eAe is eZ [[10], 3.29]

Proposition 3.17 *If A is strongly Rickart $*$- ring with center Z, then pAp has center pZ for every idempotent P.*
Proof. According to Corollary 3.9 A is an abelian. So, by ([10], 3.31), A is a compressible. pAp has center pZ for all idempotent p.

Corollary 3.18 *Every strongly Rickart $*$- ring is compressible.*

Corollary 3.19 *If A is a strongly Rickart $*$- ring, then pAp is compressible for every idempotent $P \in A$.*
Proof. follows Corollary 3.10 and ([10], 3.30).

As a similar to ([10], Proposition 1.13), the end of this paper by the following proposition.

Proposition 3.20 *Let A be a $*$- ring, then the following statements are equivalent:*

1. *A is a biregular and strongly Rickart $*$-ring*
2. *A is a $*$-biregular with proper involution $*$*
3. *For all $y \in A$, the principle ideal of yA is generated by central projection p such that $yA = pA$.*

$(1 \Rightarrow 2)$ *by [11], every biregular $*$-ring is $*$-biregular. Since, A is Rickart $*$-ring (Remark 3.2(4)). So, the involution is proper ([10], 1.10).*
$(2 \Rightarrow 1)$ *by ([11], Proposition 3), A is a biregular and so A is an abelian. Again by ([11], Proposition 3), A is a Rickart $*$-ring. So, abelian and $*$-abelian are equivalent (2.4. Proposition). Therefore, A is a strongly Rickart $*$- ring (Theorem 3.7).*
$(1 \Rightarrow 3)$ *let $y \in A$. Put $y = Ap$ for central projection p. So, $yA = yr = (1-p)A$. One put $h = 1 - p$, then, clearly that h is central projection.*
$(3 \Rightarrow 1)$ *let $y \in A$ and $yA = pA$ for central projection p then, $y = py$. $p = yx$ for $x \in A$, then $y = py = (yx)x = yx^2$. A is a biregular and $y = (yA) = (pA) = A(1-p)$. Thus, A is a strongly Rickart $*$-ring.*

References

[1] A. Hattori, A foundation of torsion theory for modules over general rings, *Nagoya Math. J.*, **17** (1960), 147-158.
https://doi.org/10.1017/s0027763000002099

[2] A. Khairnar and B. N. Waphare, Unitification of Weakly p.q.Baer *-rings, (Private communication).

[3] C. Huh, H. K. Kim, N. Y. Kim and Y. Lee, Basic examples and extensions of symmetric rings, *Journal of Pure and Applied Algebra*, **202** (2005), 154-167. https://doi.org/10.1016/j.jpaa.2005.01.009

[4] E. Akalan and L. Vas, Classes of Almost Clean Rings, *Algebra Represent. Theory*, **16** (2013), 843-857. https://doi.org/10.1007/s10468-012-9334-6

[5] G.F. Birkenmeier, J. Y. Kim and J.K. Park, Principally quasi-Baer rings, *Comm. in Algebra*, **29** (2001), 639-660. https://doi.org/10.1081/agb-100001530

[6] H. Heatherly and R. P. Tucci, Central and Semicentral Idempotents, *Kyungpook Math. J.*, **40** (2000), 255-258.

[7] J. Cirulis, Relatively Orthocomplemented Skew Nearlattices in Rickart Rings, *Demonstratio Math.*, **48** (2015), 493-508. https://doi.org/10.1515/dema-2015-0036

[8] S. A. Al-Saadi and T. A. Ibrahiem, Strongly Rickart rings, *Math. Theory and Modeling*, **4** (2014), 95-105.

[9] S. K. Berberian, *Baer *-Rings*, Springer-Verlag, New York, 1972. https://doi.org/10.1007/978-3-642-15071-5

[10] S. K. Berberian, *Baer Rings and Baer *-Rings*, The University of Texas at Austin, 2003.

[11] U. A. Aburawash and M. Saad, On Biregular IFP and Quasi-Baer -Rings, *East-West J. of Mathematics*, **16** (2014), 182-192.

[12] W. M. Fakieh, Symmetric Rings with Involutions, *British Journal of Mathematics and Computer Science*, **8** (2015), 492-505. https://doi.org/10.9734/bjmcs/2015/17267

Received: December 27, 2017; Published: January 25, 2018