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Numerical Solutions for 2D Depth-averaged Shallow Water Equations

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Abstract

This paper focuses on the development and evaluation of numerical methods for the 2D depth-averaged shallow water equations by proposing new techniques using an explicit centered finite difference and Leap-Frog schemes with Robert- Asselin filter. First, an explicit finite difference and leapfrog schemes are introduced which are effective for modelling in oceanic. Secondly, a new algorithm is proposed for 2D shallow water equations using structured grids. In order to make the model efficient and stable, a new approach is proposed for the stability analysis of structured numerical schemes for shallow water equations.

Open boundary conditions were applied at the boundaries and implement numerical simulation is conducted by computer programming using Matlab and Fortran. The performance of the proposed a new technique was tested on a number of numerical examples and applied in tsunami model. The numerical results show that the model more accurately by using these techniques.

Keywords: Explicit finite difference method, Leapfrog method, 2D Shallow water equations, Nnumerical fluid dynamics

1 Introduction

The shallow water equations (SWEs) describes the development of a hydrostatic homogeneous with constant density for an incompressible fluid in response to gravitational and rotational accelerations and they are derived from the principles of conservation of mass and conservation of momentum. This model is one of the simplest forms of the motion equations that can be used to describe the horizontal structure of an atmosphere and ocean that model the propagation of disturbances in fluids. This model is typically used to model river and lake hydrodynamics, tidal flows, tsunami propagation as well as coastal circulation [6],[1],[10],[2].

The shallow water equations are used when the horizontal scale of the flow is much smaller than the depth of the fluid. The main simplification that underlies the shallow water equations that hydrostatic balance between the gravity and pressure gradient in vertical direction implying that the vertical acceleration is negligible therefore horizontal flow is independent of height. However, the vertical velocity is not necessarily zero vertically integrating . Vertical velocity is allowed to be removed from the equations [5],[12].

Numerical methods have become well established as tools for solving shallow water equations . There have been various numerical methods to simulate the SWEs such as the finite difference scheme [3].

In fluid dynamics, fluid flow is known as the Navier-Stocks equation. The 2D shallow water models are a good approximation of the fluid motion equation when fluid density is homogeneous and depth is small in comparison to characteristic horizontal distance.

Also, shallow water equations is very commonly used for the numerical simulation of various geophysical shallow-water flows such as rivers, lakes or coastal areas, rainfall runoff from agricultural fields or even atmosphere or avalanches when completed with appropriate source terms. A numerical study of hydrodynamic surface wave propagation is a very difficult problem through the phenomena that represent (giant waves, Tsunamis, ..etc).

This paper is organized as follows: The mathematical description of the 2D depth-averaged shallow water equations are introduced in Section 2 . A new algorithm for the implementation of 2D depth-averaged shallow water equations using center finite difference and leapfrog schemes with Robert-Asselin filter has been constructed in section 3.

In section 4, Some of the numerical test are presented using the center finite difference in space and leapfrog schemes in time with Robert-Asselin filter , summary, and conclusion are given in section 5.

2 Model Description

2.1 Governing equation (Shallow water equations)

Consider the 2D depth-averaged (sometimes called depth-integrated) shallow water equations in Cartesian coordinate are obtained by integration the 3D Navier-Stokes equations over the flow depth which consists of the continuity equation and momentum conservation equations as follows:[4]

$$\frac{\partial \eta}{\partial t} + \frac{\partial Hu}{\partial x} + \frac{\partial Hv}{\partial y} = 0 \tag{1}$$

$$\frac{\partial Hu}{\partial t} + \frac{\partial Hu^2}{\partial x} + \frac{\partial Huv}{\partial y} = -gH\frac{\partial \eta}{\partial x} + \nu\left[\frac{\partial}{\partial x}(H\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(H\frac{\partial u}{\partial y})\right] + fHv + \frac{\tau_u^w}{\rho_0} - \frac{\tau_u^b}{\rho_0} \tag{2}$$

$$\frac{\partial Hv}{\partial t} + \frac{\partial Hvu}{\partial y} + \frac{\partial Hv^2}{\partial y} = -gH\frac{\partial \eta}{\partial y} + \nu\left[\frac{\partial}{\partial x}(H\frac{\partial v}{\partial x}) + \frac{\partial}{\partial y}(H\frac{\partial v}{\partial y})\right] - fHu + \frac{\tau_v^w}{\rho_0} - \frac{\tau_v^b}{\rho_0} \tag{3}$$

Here x, y are the horizontal coordinates, t is the time, u = u(x, y,t) is the depth-integrated velocity in the x direction. v = v (x, y,t) is the depth-integrated velocity in the y direction. H = H(x, y,t) is the depth from the surface level to the bottom .ν is the horizontal turbulent viscosity , g stains for the gravity acceleration , $f = 1.01 \times 10^{-4} rad/s$ is the Coriolis frequency at 42° of latitude, $\rho_0 = 1033kg/m^3$ is the water mean density , η is the water level relative to rest(water surface elevation) , τ_u^w is the bottom stress zonal component and τ_u^b is the wind stress zonal component. The above equations have the following conditions:

1. Water is incompressible.
2. Density variations are important only for buoyancy forces and considered uniform in the vertical direction.
3. Vertical accelerations can be neglected.
4. Eddy viscosity is considered to be zero.
5. Turbulent diffusion can be described by turbulent exchange coefficients.
6. Flow is quasi-hydrostatic.

The bottom stress is represents[9]:

$$\tau_u^b = \rho_0 C_D u_b \sqrt[2]{u_b^2 + v_b^2}$$

where C_D is the bottom drag coefficient and u_b and v_b are the zonal and meridional velocity bottom velocity components. The bottom drag coefficient is represents:

$$C_D = (k/ln(\frac{z_D + z_0}{z_0}))$$

where z_D is the distance to the bottom , $z_0 = 0.002m$ is a typical roughness length and the Von Karman constant is set to $k = 0.4$.

Also the wind stress is represents[9]:

$$\tau_u^w = \rho_a C_a u_b \sqrt[2]{u_b^2 + v_b^2}$$

where $\rho_a = 1.25kg/m^3$ is the air density , C_a is an air drag coefficient and u_{10} and v_{10} is the air speed at 10m above the water surface .

3 Computational algorithm

In this section, we propose a new algorithm for solving the 2D depth-averaged shallow water equations as follows:

1. Input model data and set initial data.The values $u_{i,j}^{n-1}$,$v_{i,j}^{n-1}$ and $\eta_{i,j}^{n-1}$ are known .At time t = nΔt = 0 (that is n = 0, and $t = n \triangle t$ also $u_{i,j}^{0} = v_{i,j}^{0} = 0, H_{i,j}^{0} = h_{i,j}$) on the open boundary $H_{i,j}^{0} = h_{i,j} + \eta_{i,j}^{0}$

2. Update model time to level (n + 1), so $t = (n + 1) \triangle t$. Solve the continuity equation to find η^{n+1} using $u_{i,j}^{n}$, $v_{i,j}^{n}$ and H^{n+1}.

3. Update model time to level (n + 1). Solve the momentum equations foru^{n+1} and v^{n+1} using H^{n+1} .

4.Apply Robert-Asselien filter for u,v andη for each time step.

5. Return to step 2 and continue until the period of the simulation is completed.

3.1 Accuracy and stability(The CFL Condition)

The Courant number CFL condition for depth-averaged 2D shallow water equations is defined as follows:[7]

$$C = \triangle t \left(\sqrt{gH} + V_{max} \right) \left(\frac{1}{\triangle x^2} + \frac{1}{\triangle y^2} \right)^{\frac{1}{2}}$$

Using the stability condition CFL< 1 in above equation .The following stability criterion is obtained for the optimal time step

$$\triangle t \leq \frac{\triangle x \triangle y}{\left(\sqrt{gH} + V_{max} \right) \sqrt{(\triangle x^2 + \triangle y^2)}}$$

- Where $c = \sqrt{gH}$ is the magnitude of the velocity (whose dimension is length/time).
- Δt is the time step (whose dimension is time).
- Δx is the length interval (whose dimension is length).

4 Results examples

In this section, we discuss the results of some examples for the 2D depth-averaged non-linear shallow water equations using an explicit finite difference and leapfrog schemes with Robert-Asselin filtering in time at different cases.

4.1 Example 1

In this example, we use an explicit center finite difference scheme in space and leapfrog scheme in time with Roberts - Asselin filtering to approximate a reduced gravity 2D depth-averaged nonlinear equations (1-3) in a rectangular domain Ω which has no bottom stress, wind stress, f=0 and viscosity horizontal in x-axis and y-axis =0 by using a Cartesian coordinate system,Ω reads $\Omega = [0, L_x] \times [0, L_y]$

Initial condition:

Initially, the water is at rest with a water drop of 10 and a zero flow everywhere. i.e no slip boundary conditions: u = v = 0 on $\partial\Omega$ and

$$\eta(i, j) = 10 * exp((-5((x)^2 + (y)^2))$$

$$u(x, y, t = 0) = 0, v(x, y, t = 0) = 0$$

Boundary condition:

Here, Reflexive boundary conditions were implemented at the boundaries with CFL condition 0.13

Numerical parameters:

The computational domain is discretized by a grid whosesize is regular. Numerical values of the parameters are chosen as follows: $L_x = L_y = 200$, dx=0.10 , dy=0.10 (grid length) and the time steps $\triangle t$=0.01 sec at time t=50,100,200,...,1000 hours .This model is discretized on an Arakawa C grid , the water height η being located at the center of the cells and the velocity components at the center of the cell edges.

Results and discussion:

First of all, we tested the computational stability and accuracy of this system of equations.The time integrations were performed for 10000 hours, both with an explicite center finite difference and leap-frog schemes ($\triangle t = 0.01 sec$). The calculations were stable

The following figures show the approximate solution , the global relative error and compare the solution for the free surface, u-velocity and v-velocity for the 2D nonlinear shallow water equations.

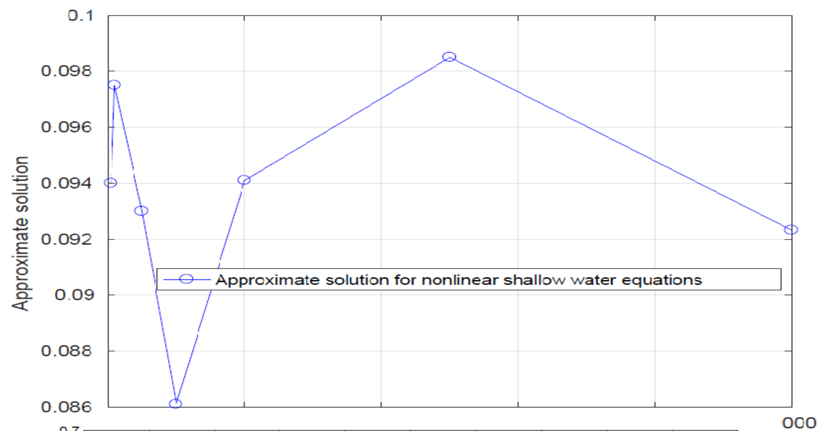

Figure ations

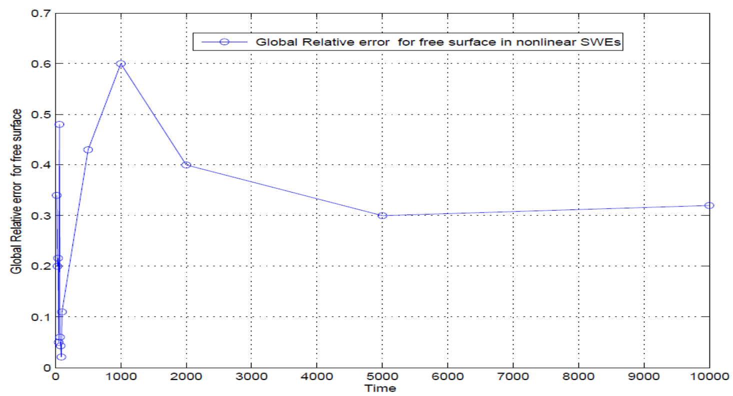

Figure 2: Global relative error for the free surface in nonlinear SWEs

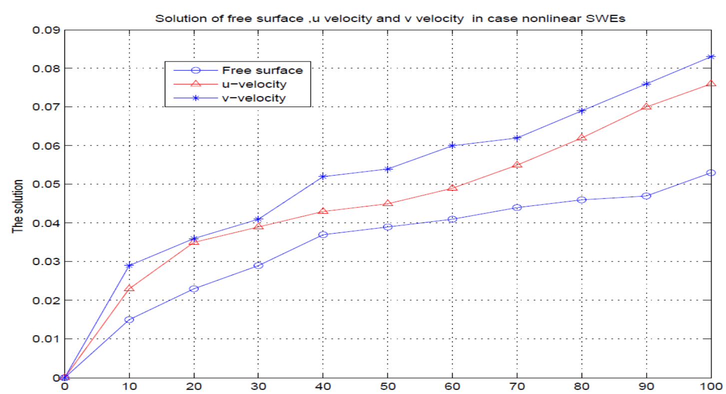

Figure 3: Solution for the free surface ,u velocity and v velocity in nonlinear SWEs at 100 days

4.2 Example 2 :

In this example, we use an explicit center finite difference scheme in space and leapfrog scheme in time with Asselin-Roberts filtering to approximate a reduced gravity 2D depth-averaged linear equations in a rectangular domain

Ω, which has the terms $\frac{\partial u^2}{\partial x} = 0$ $\frac{\partial v^2}{\partial y} = 0$, $\frac{\partial uv}{\partial y} = 0$, $\frac{\partial uv}{\partial x} = 0$, $\frac{\partial u^2}{\partial y} = 0$ $\frac{\partial v^2}{\partial x} = 0$ and f=0, using a Cartesian coordinate system,Ω reads $\Omega = [0, L_x] \times [0, L_y]$

Initial condition:

Initially the water is at rest with a water drop of 1.6 and a zero flow everywhere.

$$\eta(i, j) = 10 * exp((-((i - i_0)^2 + (j - j_0)^2))/(k^2))$$

Where i_0=15, j_0=15 and k=6

Boundary condition:

Here,Dirchelet boundary conditions were implemented at the boundaries with CFL condition 0.13

Numerical parameters:

The computational domain is discretized by a grid whose size is regular. Numerical values of the parameters are chosen as follows: $L_x = L_y = 200$, dx=0.10 , dy=0.10 (grid length) and the time steps $\triangle t = 0.01s$ at time t=100,200...hours..

Results and discussion:

First of all,we tested the computational stability and accuracy of this system of equations.The time integrations were performed for 1000 hours, both with an explicite center finite difference and leap-frog schemes ($\triangle t = 0.01sec$). The calculations were stable

The following figures show the solution for free surface , global relative error and compare the approximate solution for free surface,u-velocity and v-velocity in 2D linear shallow water equations

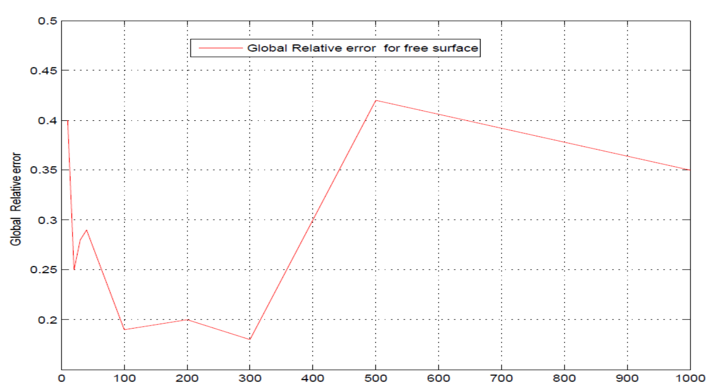

Figure 4: Global relative error for the free surface in linear SWEs

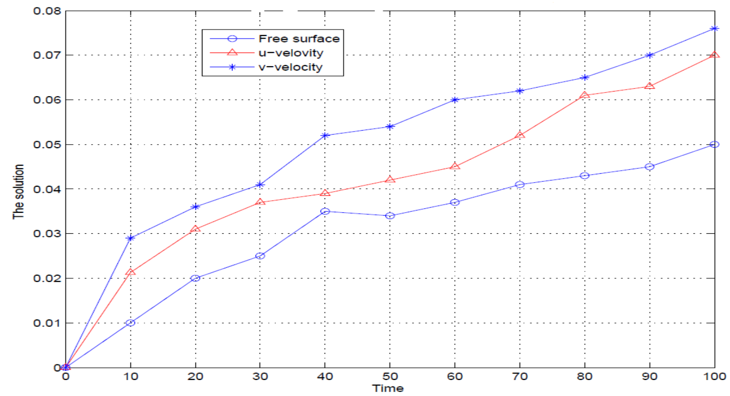

Figure 5: Compare the approximate solution for free surface in linear SWEs at 100 days

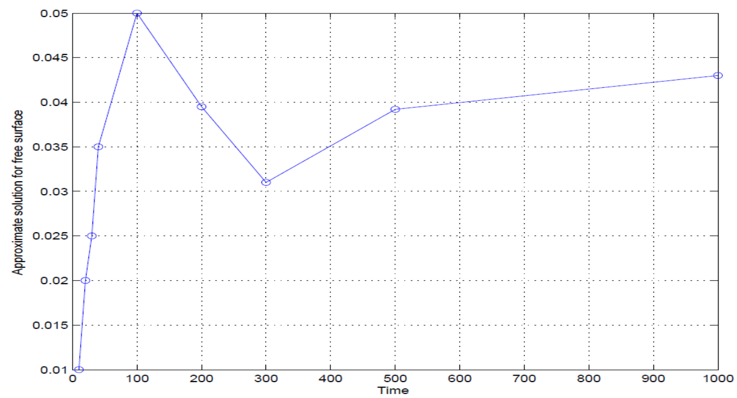

Figure 6: Approximate solution in linear SWEs

4.3 Application for the 2D depth-averaged non - linear shallow water equations

In this section, we introduce some applications for the 2D depth-averaged non-linear shallow water equations obtained by considerning the Reynolds averaged 3D Navier-Stokes equations for incompressible fluid neglecting viscosity , wind stresses and the coriolis force f terms using center finite difference scheme in space and leapfrog scheme in time with Robert- Assielin filter to approximate this system .

Initial condition:

Initially, it is assumed that the motion in the domain is observed from an initial state of rest, so $u(x, y, 0) = v(x, y, 0) = 0$ and at the begining of a simulation start, this initial water surface displacement is intrpolated into all sub-level grids([11],[8]).

Boundary condition:

Here, radaition open boundary conditions were implemented at the boundaries with CFL condition 0.7.

Example 1 :

In this example, we use system of 2D depth- averged linear shallow water equations(we can called system of 2D reduced gravity) with wind stress= 0 and non-rotated f=0 at time t= 1000, 2000,..., 5000 min and the numerical values of the parameters are chosen as follows: $L_x = L_y = 200$, nx=120, ny=120, dx=9, dy=9 (grid length size), and time steps $t = 2.5e^{-2}$, wave lenght .5m and water depth 10m.

First of all,we tested the computational stability and accuracy of this system of equations.The time integrations were performed for 5000 min both with an explicit center finite difference and leap-frog schemes ($\triangle t = 2.5e^{-2}$). The calculations were stable

The first figure compares the approximate solution for the free surface,u-velocity and v-velocity and the second figure shows relative error l2 in 2D linear shallow water equations .

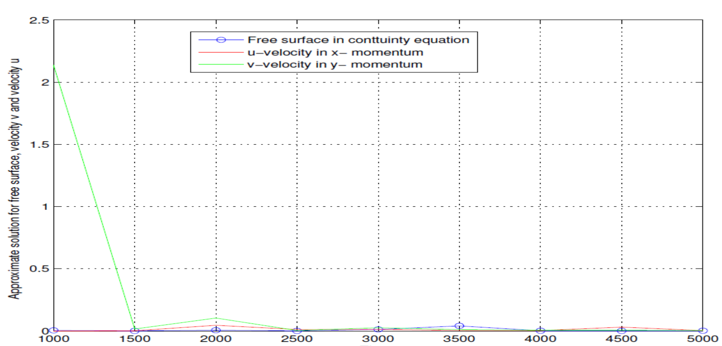

Figure 7: Compare the approximate solution between free surface , u-velocity and v-velocity in linear SWEs

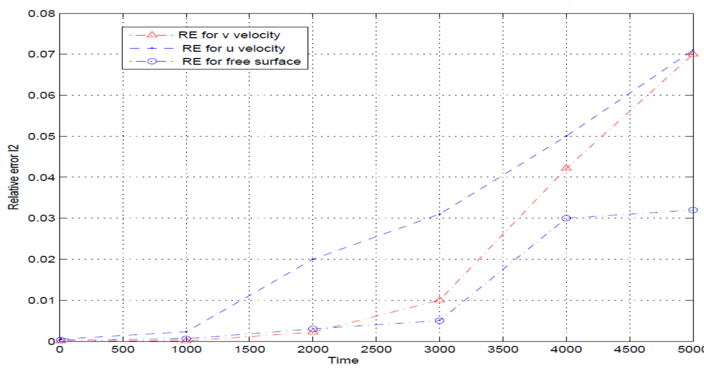

Figure 8: Compare the relative errorl2 for the free surface,u-velocity and v-velocity in linear SWEs

Example 2:

In this example, we introduce some application for the 2D depth-averaged nonlinear shallow water equations obtained by considering the Reynolds averaged Navier-Stokes equations for incompressible fluid neglecting viscosity, wind stresses, and the Coriolis force f terms. For this example, we use the center finite difference scheme in space and leapfrog scheme in time with Robert-Asselin filtering to approximate this system.

Initial condition:

Initially, it is assumed that the motion in the domain is observed from an initial state of rest, so $u(x, y, 0) = v(x, y, 0) = 0$ and at the begining of a simulationstart, this initial water surface displacement is intrpolated into all sub-level grids.A full description of initial, boundary condition and model configuration can be found in ([11],[8]).

Boundary condition:

Here, radaition open boundary conditions were implemented at the boundaries with CFL condition 0.7. In this example, we use system of 2D depth-averged nonlinear shallow water equations ($\nu = 0$, wind stress =0, non-rotated f=0) at time t= 1000, 2000,...,5000 min and the numerical values of the parameters are chosen as follows: nx=120, ny=120, dx=9, dy=9 (grid length size), and time steps $t = 2.5e^{-2}$, wave lenght .5m and water depth 10m.

The first figure compares the approximate solution for the free surface,u-velocity and v-velocity and the second figure shows the relative errorl2 in 2D nonlinear shallow water equations.

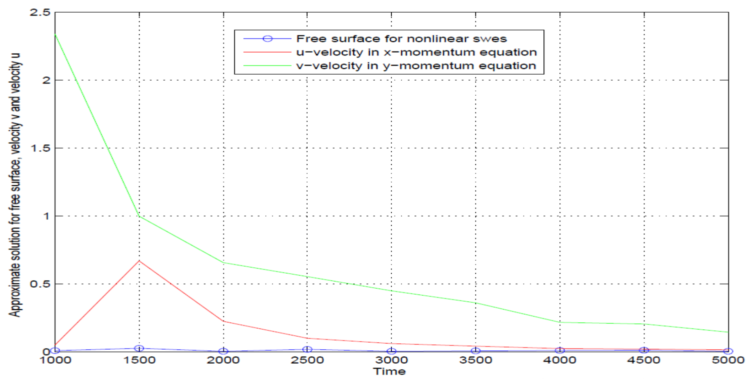

Figure 9: Compare approximate solution between free surface , u-velocity and v-velocity in nonlinear SWEs

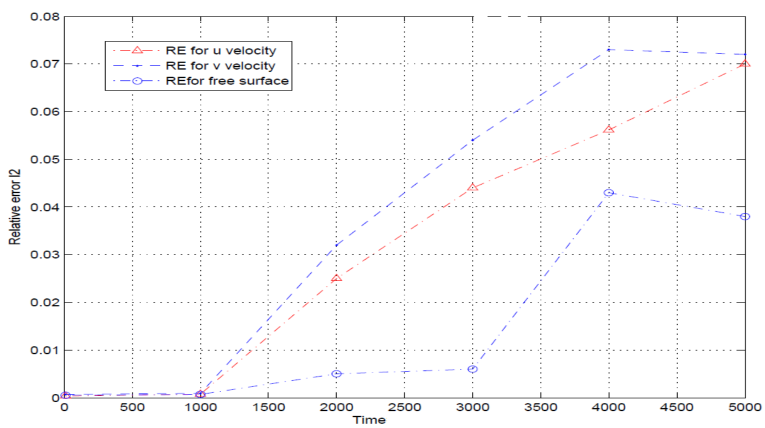

Figure 10: Compare the relative error l2 between free surface , u-velocity and v-velocity in nonlinear SWEs

4.4 Sumary and conclusion

In our work, we assessed the performance of a new technique for 2D shallow water equation, and we have implemented a new algorithm using center finite difference and Leapfrog scheme with Asselin-Robert filtering by using the Dirichlet open boundary conditions.

The performance of the model has tested some examples of the tsunami model and results generated were of high accuracy, with the global relative errorl2 . The results demonstrate the applicability and benefits of this technique.

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