Diophantine Equations. Elementary Methods

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Abstract

In this note we are interested in some diophantine equations to the form
\[ \sum_{j=1}^{h} k_j x_j^{r_j} = k_{h+1} x_{h+1}^{r_{h+1}}. \]
Some of these diophantine equations are well-known and our methods of solution are different and very elementary.

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1 Introduction and Main Results

In this note we are interested in some diophantine equations to the form
\[ \sum_{j=1}^{h} k_j x_j^{r_j} = k_{h+1} x_{h+1}^{r_{h+1}} \tag{1} \]
where \( h \geq 2 \), the coefficients \( k_j \) \((j = 1, \ldots, h + 1)\) are integers different of zero and the exponents \( r_j \geq 2 \) \((j = 1, \ldots, h + 1)\) are positive integers.

Some of these diophantine equations are well-known (see [1] and [2]) and our methods of solution are different and very elementary.

Let us consider a solution
\[ (x_1, x_2, \ldots, x_h, x_{h+1}) \tag{2} \]
to equation (1) (the \( x_j \) \((j = 1, \ldots, h + 1)\) are integers). If we multiply both sides of equation (1) by \( E^L \), where \( E \) is an integer different of zero and \( L \) is
the least common multiple ($lcm$) of the exponents $r_j$ ($j = 1, \ldots, h + 1$), then we obtain the solution

$$
\left(x_1 E^{\frac{r_1}{h}}, x_2 E^{\frac{r_2}{h}}, \ldots, x_h E^{\frac{r_h}{h}}, x_{h+1} E^{\frac{r_{h+1}}{h}}\right)
$$

(3)

The solution (3) will be called derivada solution of (1). Note that solution (2) is derivada solution of solution (2) if we put $E = 1$. Clearly, from (3) we can obtain (2) by common factor. If a set of solutions of equation (1) contain at least one derivada solution of each solution of equation (1) we shall call this set of solutions a complete system of solutions to equation (1). Note that from a complete system of solutions we can obtain all solution to the equation by common factor. This method if not very different to consider the set of primitive solutions to, for example, the equation $x^2 + y^2 = z^2$ and to obtain the rest of the solutions by multiplication of the primitive solutions.

If we consider a certain subset $S$ of solutions to equation (1) then a complete system of solutions in relation to $S$ is a subset of $S$ that contain at least a derivada solution of each solution of the set $S$.

We begin with the famous Pythagorean equation.

**Theorem 1.1** Let us consider the diophantine equation

$$
x^2 + y^2 = z^2
$$

(4)

where $xyz \neq 0$. Then, a complete system of solutions to the equation is

$$
x = a^2 - b^2, \quad y = 2ab, \quad z = -a^2 - b^2
$$

(5)

where $a$ and $b$ are arbitrary integers such that $xyz \neq 0$.

Proof. This equation has solutions $(x, y, z)$ such that $xyz \neq 0$. For example $(x, y, z) = (3, 4, 5)$. Let us consider then a solution $(x, y, z)$ such that $xyz \neq 0$. We can write

$$
(x, y, z) = (C + a, b, C)
$$

(6)

Note that $a \neq 0$. Consequently (see (4))

$$(C + a)^2 + b^2 = C^2$$

(7)

Therefore

$$
C = -\frac{a^2 + b^2}{2a}
$$

(8)
Substituting (8) into (7) we obtain

\[ \left( \frac{-a^2 + b^2}{2a} + a \right)^2 + b^2 = \left( \frac{-a^2 + b^2}{2a} \right)^2 \]  

(9)

If we now multiply both sides of equation (9) by \((2a)^2\) then we obtain the derivada solution (5)

\[ (a^2 - b^2)^2 + (2ab)^2 = (-a^2 - b^2)^2 \]  

(10)

of the solution \((x, y, z)\). Note, besides, that (10) is an identity. The theorem is proved.

**Theorem 1.2** Let us consider the diophantine equation

\[ x_1^2 + \sum_{j=2}^{h} k_j x_j^2 = x_{h+1}^2 \]  

(11)

where \(h \geq 2\), the coefficients \(k_j\) \((j = 2, \ldots, h)\) are positive integers and some \(x_j\) \((j = 2, \ldots, h)\) is different of zero. Then a complete system of solutions to the equation is

\[ x_1 = a_1^2 - \sum_{j=2}^{h} k_j a_j^2, \quad x_j = 2a_1 a_j \quad (j = 2, \ldots, h), \quad x_{h+1} = -a_1^2 - \sum_{j=2}^{h} k_j a_j^2 \]  

(12)

where the \(a_j\) \((j = 1, \ldots, h)\) are arbitrary integers such that some \(x_j\) \((j = 2, \ldots, h)\) is different of zero.

Proof. The equation has solutions with is property, since we have the identity (see (11))

\[ \left( a_1^2 - \sum_{j=2}^{h} k_j a_j^2 \right)^2 + \sum_{j=2}^{h} k_j (2a_1 a_j)^2 = \left( -a_1^2 - \sum_{j=2}^{h} k_j a_j^2 \right)^2 \]  

(13)

Let us consider then a solution \((x_1, \ldots, x_h, x_{h+1})\) with is property. We can write

\[ (x_1, x_2, \ldots, x_h, x_{h+1}) = (C + a_1, a_2, \ldots, a_h, C) \]  

(14)

Note that \(C \neq 0\) and \(a_1 \neq 0\), since in contrary case the property is not fulfilled. Consequently (see (11))

\[ (C + a_1)^2 + \sum_{j=2}^{h} k_j a_j^2 = C^2 \]  

(15)
Therefore

\[ 2Ca_1 + a_1^2 + \sum_{j=2}^{h} k_ja_j^2 = 0 \]

That is

\[ C = -\frac{a_1^2 + \sum_{j=2}^{h} k_ja_j^2}{2a_1} \] (16)

Substituting (16) into (15) we obtain

\[ \left( -\frac{a_1^2 + \sum_{j=2}^{h} k_ja_j^2}{2a_1} + a_1 \right)^2 + \sum_{j=2}^{h} k_ja_j^2 = \left( -\frac{a_1^2 + \sum_{j=2}^{h} k_ja_j^2}{2a_1} \right)^2 \] (17)

If we now multiply both sides of equation (17) by \((2a_1)^2\) then we obtain the derivada solution (12)

\[ \left( a_1^2 - \sum_{j=2}^{h} k_ja_j^2 \right)^2 + \sum_{j=2}^{h} k_j(2a_1a_j)^2 = \left( -a_1^2 - \sum_{j=2}^{h} k_ja_j^2 \right)^2 \] (18)

of the solution \((x_1, \ldots, x_h, x_{h+1})\). The theorem is proved.

**Theorem 1.3** Let us consider the diophantine equation

\[ \sum_{j=1}^{h} k_jx_j^2 = k_{h+1}x_{h+1}^2 \] (19)

where \(h \geq 2\) and the coefficients \(k_j (j = 1, \ldots, h)\) and \(k_{h+1}\) are positive integers. Suppose that this equation has a solution

\[ (x_1, x_2, \ldots, x_h, x_{h+1}) = (b_1, b_2, \ldots, b_h, b_{h+1}) \] (20)

different of the trivial solution \((0, 0, \ldots, 0, 0)\) and besides \(\gcd(b_1, b_2, \ldots, b_h, b_{h+1}) = 1\). Then a complete system of solutions is

\[ x_j = -b_j \sum_{i=1}^{h} k_ic_i^2 + 2c_j \sum_{i=1}^{h} k_ib_ic_i \quad (j = 1, 2, \ldots, h) \] (21)

\[ x_{h+1} = -b_{h+1} \sum_{i=1}^{h} k_ic_i^2 \] (22)

where the \(c_i (i = 1, \ldots, h)\) are arbitrary integers.
Proof. Let us consider a solution to equation (19) \((e_1, e_2, \ldots, e_h, C')\) where the \(e_j\) \((j = 1, \ldots, h)\) and \(C'\) are integers and \(C' \neq 0\). Suppose that this solution can not be written in the form \((b_1 C, b_2 C, \ldots, b_h C, b_{h+1} C)\) where \(C\) is a integer different of zero.

Solutions with this property exist, since we have the identity (compare with (21) and (22))

\[
\sum_{j=1}^{h} k_j \left( -b_j \sum_{i=1}^{h} k_i c_i^2 + 2c_j \sum_{i=1}^{h} k_i b_i c_i \right)^2 = k_{h+1} \left( -b_{h+1} \sum_{i=1}^{h} k_i c_i^2 \right)^2
\]

We can write

\[
(e_1, e_2, \ldots, e_h, C') = (b_1 C + a_1, b_2 C + a_2, \ldots, b_h C + a_h, b_{h+1} C)
\]

Consequently

\[
C = \frac{C'}{b_{h+1}}
\]

and

\[
a_j = e_j - \frac{b_j}{b_{h+1} C'} = \frac{c_j}{b_{h+1}} \quad (j = 1, \ldots, h)
\]

where the \(c_j\) \((j = 1, \ldots, h)\) are integers. Note that some \(a_j\) is different of zero and consequently some \(c_j\) is different of zero (see (26)), since in contrary case we have (see (24))

\[
(e_1, e_2, \ldots, e_h, C') = (b_1 C, b_2 C, \ldots, b_h C, b_{h+1} C)
\]

where \(C\) is given by (25). This is impossible, \(C\) can not be a rational not integer since \(\gcd(b_1, b_2, \ldots, b_h, b_{h+1}) = 1\) and \(C\) can not be a integer by the established property of the solution (see above).

Substituting (25) and (26) into (24) we obtain

\[
(e_1, e_2, \ldots, e_h, C') = \left( \frac{b_1}{b_{h+1}} C' + \frac{c_1}{b_{h+1}}, \frac{b_2}{b_{h+1}} C' + \frac{c_2}{b_{h+1}}, \ldots, \frac{b_h}{b_{h+1}} C' + \frac{c_h}{b_{h+1}}, C' \right)
\]

Substituting this solution into equation (19) we have

\[
\sum_{j=1}^{h} k_j \left( \frac{b_j}{b_{h+1}} C' + \frac{c_j}{b_{h+1}} \right)^2 = k_{h+1} \left( C' \right)^2
\]

That is (use \(\sum_{j=1}^{h} k_j b_j^2 = k_{h+1} b_{h+1}^2\) (see (20)))

\[
C' \left( \sum_{j=1}^{h} 2k_j b_j c_j \right) + \sum_{j=1}^{h} k_j c_j^2 = 0
\]
That is

$$C' = \frac{-\sum_{j=1}^{h} k_j c_j^2}{\sum_{j=1}^{h} 2k_j b_j c_j}$$

(29)

since \(\sum_{j=1}^{h} k_j c_j^2 \neq 0\) and consequently (see (28)) \(\sum_{j=1}^{h} 2k_j b_j c_j \neq 0\).

Substituting (29) into (27) we obtain

$$\sum_{j=1}^{h} k_j \left( \frac{b_j - \sum_{i=1}^{h} h_i c_i^2 + c_j}{b_{h+1} \sum_{i=1}^{h} 2k_i b_i c_i} \right)^2 = k_{h+1} \left( \frac{-\sum_{i=1}^{h} k_i c_i^2}{\sum_{i=1}^{h} 2k_i b_i c_i} \right)^2$$

(30)

If we now multiply both sides of (30) by \(b_{h+1}^2 (\sum_{i=1}^{h} 2k_i b_i c_i)^2\) then we obtain the following derivada solution

$$\sum_{j=1}^{h} k_j \left( -b_j \sum_{i=1}^{h} k_i c_i^2 + 2c_j \sum_{i=1}^{h} k_i b_i c_i \right)^2 = k_{h+1} \left( -b_{h+1} \sum_{i=1}^{h} k_i c_i^2 \right)^2$$

(31)

of the solution \((e_1, e_2, \ldots, e_h, C')\) (see (31), (21) and (22)).

Suppose now that \((e_1, e_2, \ldots, e_h, C') = (b_1 C, b_2 C, \ldots, b_h C, b_{h+1} C)\) where \(C\) is an integer different of zero. Suppose that some \(b_s = 0\), then we put into (21) and (22) \(c_s = C\) and \(c_j = 0\ (j \neq s)\) and obtain la derivada solution

$$x_j = -b_j k_s C^2 = b_j C(-k_s C) \quad (j = 1, 2, \ldots, h)$$

$$x_{h+1} = -b_{h+1} k_s C^2 = b_{h+1} C(-k_s C)$$

Suppose now that \(b_j \neq 0\ (j = 1, \ldots, h)\), then we put into (21) and (22) \(c_1 = -k_2 b_2 C, c_2 = k_1 b_1 C\) and \(c_j = 0\ (j \neq 1, 2)\) and obtain the derivada solution

$$x_j = -b_j (k_1 c_1^2 + k_2 c_2^2) = -b_j (k_1 (-k_2 b_2)^2 + k_2 (k_1 b_1)^2) C^2 \quad (j = 1, 2, \ldots, h)$$

$$x_{h+1} = -b_{h+1} (k_1 c_1^2 + k_2 c_2^2) = -b_{h+1} (k_1 (-k_2 b_2)^2 + k_2 (k_1 b_1)^2) C^2$$

If \((e_1, e_2, \ldots, e_h, C') = (0, 0, \ldots, 0, 0)\) then we put \(c_j = 0\ (j = 1, \ldots, h)\) into (21) and (22) and obtain the derivada solution \((0, 0, \ldots, 0, 0)\). The theorem is proved.

**Example 1.4** Let us consider the diophantine equation (Legendre’s equation)

$$k_1 x_1^2 + k_2 x_2^2 = k_3 x_3^2$$
Suppose that \((b_1, b_2, b_3)\) is a solution different of \((0, 0, 0)\) and \(\gcd(b_1, b_2, b_3) = 1\). Then a complete system of solutions is

\[
x_1 = -b_1(k_1c_1^2 + k_2c_2^2) + 2c_1(k_1b_1c_1 + k_2b_2c_2)
\]

\[
x_2 = -b_2(k_1c_1^2 + k_2c_2^2) + 2c_2(k_1b_1c_1 + k_2b_2c_2)
\]

\[
x_3 = -b_3(k_1c_1^2 + k_2c_2^2)
\]

where \(c_1, c_2\) and \(c_3\) are arbitrary integers.

**Theorem 1.5** Let us consider the diophantine equation

\[
\sum_{j=1}^{h} k_j x_j^{r_j} = k_{h+1}^{M+1} x_{h+1}^{M+1}
\]

where \(h \geq 2\), the coefficients \(k_j\) (\(j = 1, \ldots, h\)) and \(k_{h+1}\) are integers different of zero and each integer exponent \(r_j \geq 2\) (\(j = 1, \ldots, h\)) divides the positive integer \(M\). Let us consider the solutions to the equation

\[
(x_1, \ldots, x_h, x_{h+1})
\]

where \(x_{h+1} \neq 0\). Then a complete system of solutions to the equation is

\[
x_j = k_{h+1}^{r_j} A b_j^{r_j} \quad (j = 1, 2, \ldots, h)\]

\[
x_{h+1} = k_{h+1}^{M-1} A
\]

where

\[
A = \sum_{i=1}^{h} k_i b_i^{r_i}
\]

and the \(b_j\) are arbitrary integers such that \(A \neq 0\).

Proof. We have the identity (compare with (34) and (35))

\[
\sum_{j=1}^{h} k_j \left( k_{h+1}^{r_j} A b_j^{r_j} \right) = k_{h+1} \left( k_{h+1}^{M-1} A \right)^{M+1}
\]

where

\[
A = \sum_{j=1}^{h} k_j b_j^{r_j} \neq 0
\]
Consequently the diophantine equation (32) has infinite solutions where \( x_{h+1} \neq 0 \). Now, we shall prove that if
\[
(x_1, \ldots, x_h, x_{h+1})
\]
is a solution with \( x_{h+1} \neq 0 \) then there exist integers \( b_j (j = 1, \ldots, h) \) such that equations (34) and (35) are a derivada solution of the solution (39). Thus, equations (34) and (35) with the condition \( A \neq 0 \) are a complete system of solutions to the equation. Therefore, let us consider a solution to the equation
\[
(x_1, \ldots, x_h, x_{h+1}) = (b_1, \ldots, b_h, C)
\]
where \( C \neq 0 \). Therefore we have
\[
\sum_{j=1}^{h} k_j b_j^{r_j} = k_{h+1} C^{M+1}
\]
(41)
If now we multiply both sides of (41) by
\[
\left( k_{h+1}^{M} \right)^{M+1} (C^{M})^{M+1} = \left( k_{h+1}^{M+1} \right) (C^{M+1})^{M}
\]
then we obtain the derivada solution
\[
\sum_{j=1}^{h} k_j \left( k_{h+1}^{M+1} C^{M+1} \right)^{r_j} = k_{h+1} \left( k_{h+1}^{M+1} C^{M+1} \right)^{M+1}
\]
(42)
Equation (41) gives (see (42))
\[
x_{h+1} = k_{h+1}^{M+1} = k_{h+1}^{M-1} \left( \sum_{j=1}^{h} k_j b_j^{r_j} \right) = k_{h+1}^{M-1} A
\]
(43)
Therefore (see (43) and (42)) we have
\[
x_j = \left( k_{h+1}^{M-1} A \right)^{r_j} k_{h+1}^{M} b_j = k_{h+1}^{M} A^{r_j} b_j \quad (j = 1, 2, \ldots, h)
\]
(44)
Consequently equations (43) and (44) are a derivada solution of the solution (40). The theorem is proved.

**Example 1.6** Now, we give some examples of Theorem 1.5. The equations in a), b) and c) are consider in [2].
a) The equation \( x^2 + 3y^2 = z^3 \).
This equation has the following complete system of solutions.
\[
x = a(a^2 + 3b^2), \quad y = b(a^2 + 3b^2), \quad z = a^2 + 3b^2
\]
where \(a\) and \(b\) are arbitrary integers.

b) The equation \(x^2 + 3y^2 = 4z^3\). This equation has the following complete system of solutions.
\[
x = 16a(a^2 + 3b^2), \quad y = 16b(a^2 + 3b^2), \quad z = 4(a^2 + 3b^2)
\]
where \(a\) and \(b\) are arbitrary integers.

c) The equation \(x^4 + 3y^4 = z^5\). This equation has the following complete system of solutions.
\[
x = a(a^4 + 3b^4), \quad y = b(a^4 + 3b^4), \quad z = a^4 + 3b^4
\]
where \(a\) and \(b\) are arbitrary integers.

d) The equation \(3x^3 - 2y^4 + z^5 = w^{121}\). This equation has the following complete system of solutions.
\[
x = a(3a^3 - 2b^4 + c^5)^{40}, \quad y = b(3a^3 - 2b^4 + c^5)^{30}
\]
\[
z = c(3a^3 - 2b^4 + c^5)^{24} \quad w = 3a^3 - 2b^4 + c^5
\]
where \(a\), \(b\) and \(c\) are arbitrary integers.

e) The equation \(4x^2 + 3y^4 + 2z^8 = w^9\). This equation has the following complete system of solutions.
\[
x = a(3a^3 - 2b^4 + c^5)^{40}, \quad y = b(3a^3 - 2b^4 + c^5)^{30}
\]
\[
z = c(3a^3 - 2b^4 + c^5)^{24} \quad w = 3a^3 - 2b^4 + c^5
\]
where \(a\), \(b\) and \(c\) are arbitrary integers.

f) The equation \(x_1^n + x_2^n + \cdots + x_h^n = x_{h+1}^{n+1}, \ (x_{h+1} \neq 0)\). This equation has the following complete system of solutions.
\[
x_j = b_j \left( \sum_{i=1}^{h} b_i^n \right) \quad (j = 1, \ldots, h), \quad x_{h+1} = \sum_{i=1}^{h} b_i^n \quad (45)
\]
where \(b_1, b_2, \ldots, b_h\) are arbitrary integers.

In particular, the equation \(x^n + y^n = z^{n+1}\) has the following complete system of solutions.
\[
x = a(a^n + b^n), \quad y = b(a^n + b^n), \quad z = a^n + b^n \quad (46)
\]
where \(a\) and \(b\) are arbitrary integers.

Now, we generalize the former theorem.
Theorem 1.7 Let us consider the diophantine equation
\[ \sum_{j=1}^{h} k_j x_j^{r_j} = k_{h+1} x_{h+1}^{\frac{M+1}{d}} \]
where \( h \geq 2 \), the coefficients \( k_j \) (\( j = 1, \ldots, h \)) and \( k_{h+1} \) are integers different of zero, each integer exponent \( r_j \geq 2 \) (\( j = 1, \ldots, h \)) divides the positive integer \( M \) and the positive integer \( d \) (\( 0 < d < M+1 \)) divides \( M+1 \). Let us consider the solutions to the equation \((x_1, \ldots, x_h, x_{h+1})\) where \( x_{h+1} \neq 0 \). Then a complete system of solutions to the equation is
\[ x_j = \frac{M^2}{r_j} A_{r_j} b_j \quad (j = 1, 2, \ldots, h) \quad x_{h+1} = \left( k_{h+1}^{M-1} A \right)^d \]
where \( A = \sum_{i=1}^{h} k_i b_i^{r_i} \) and the \( b_j \) are arbitrary integers such that \( A \neq 0 \).

Proof. The proof is the same as the former theorem. The theorem is proved.

Remark 1.8 Note that the former theorem is a particular case of this theorem when \( d = 1 \).

Remark 1.9 Note that \((M+1)/d\) can be any exponent relatively prime with \( L \), where \( L \) is the least common multiple of the exponents \( r_j \) (\( j = 1, \ldots, h \)). Since the linear diophantine equation \( \frac{M+1}{d} y_1 - L y_2 = 1 \) have infinite solutions with \( y_1 > 0 \) and \( y_2 > 0 \), we take \( M = L y_2 \).

Example 1.10 Now, we give an example of Theorem 1.7. Let us consider the diophantine equation \( \sum_{j=1}^{h} k_j x_j^{r_j} = k_{h+1} x_{h+1}^2 \) where the exponents \( r_j \) (\( j = 1, \ldots, h \)) are odd. If we take \( M = \text{lcm}(r_1, r_2, \ldots, r_h) \) then a complete system of solutions to the equation is
\[ x_j = \frac{M^2}{r_j} A_{r_j} b_j \quad (j = 1, 2, \ldots, h) \quad x_{h+1} = \left( k_{h+1}^{M-1} A \right)^{\frac{M+1}{d}} \]
where \( A = \sum_{i=1}^{h} k_i b_i^{r_i} \), and the \( b_j \) are arbitrary integers such that \( A \neq 0 \). Particular cases of this example appear in [2], for example, \( x^3 + y^3 = z^2 \), \( x^3 + y^3 = 2z^2 \), \( x^3 - 2y^3 = z^2 \) and another.

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