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On a New Conjecture of Prime Numbers

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Abstract

Let p_n be the n-th prime number. In this article we prove that the Farhadian's conjecture, namely

$$p_n^{\left(\frac{p_{n+1}}{p_n}\right)^n} \leq n^{p_n}$$

holds for almost all n

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1 Introduction

Let p_n be the n-th prime. There are strong conjectures on consecutive primes, the Firoozbakht's conjecture, the Nicholson's conjecture, the Cramer's conjecture, etc. For example, the Firoozbakht's conjecture establish that the sequence $p_n^{1/n}$ is strictly decreasing for $n \geq 1$. That is

$$p_n^{\frac{1}{n}} > p_{n+1}^{\frac{1}{n+1}}$$

Recently, in 2016 (see C. Rivera, Conjecture 78, 2016, available at *http* : *//www.primepuzzles.net/conjectures/conj*078*.htm*), R. Farhadian established the following strong conjecture

$$p_n^{\left(\frac{p_{n+1}}{p_n}\right)^n} \leq n^{p_n} \tag{1}$$

for $n \geq 4$ and he proved that his conjecture is more strong than the Firoozbakht's conjecture and other strong conjectures as the Nicholson's conjecture. Besides R. Farhadian has confirmed his own conjecture for the first 10^4 primes.

Clearly, the Farhadian's conjecture is equivalent to the establishment

$$\log p_{n+1} - \log p_n \leq \frac{\log p_n}{n} - \frac{\log\log p_n - \log\log n}{n} \tag{2}$$

Now, the Firoozbakht's conjecture is equivalent to the establishment (see [1])

$$\log p_{n+1} - \log p_n < \frac{\log p_n}{n} \tag{3}$$

Consequently (2) and (3) give another proof that the Farhadian's conjecture is more strong that the Firoozbakht's conjecture.

The Farhadian's conjecture is also equivalent to the establishment

$$d_n = p_{n+1} - p_n \leq p_n \left(\left(\frac{p_n \log n}{\log p_n} \right)^{\frac{1}{n}} - 1 \right)$$

In a previous article [1] was proved that the Firoozbakht's conjecture holds for almost all n. In this article we prove that for very general sequences the Farhadian's conjecture holds for almost all n. In particular this is also true for the sequence of prime numbers.

2 Main Results

We have the following definition. This definition appear in [2].

Definition 2.1 *Let $f(x)$ be a function defined on interval $[a, \infty)$ such that $f(x) > 0$, $\lim_{x\to\infty} f(x) = \infty$ and with continuous derivative $f'(x) > 0$. The function $f(x)$ is of slow increase if and only if the following condition holds*

$$\lim_{x\to\infty} \frac{f'(x)}{\frac{f(x)}{x}} = 0 \tag{4}$$

Typical functions of slow increase are $f(x) = \log x$, $f(x) = \log^2 x$, $f(x) = \log\log x$, $f(x) = \log\log\log x$, $f(x) = \frac{\log x}{\log\log x}$, etc.

Let us consider a strictly increasing sequence of positive integers A_n such that

$$A_n \sim n^s f(n) \tag{5}$$

where $s \geq 1$ (see [2, Remark 12]) is a fixed real number and $f(x)$ is a function of slow increase with decreasing derivative (see [2, Section 2]).

Lemma 2.2 *Let us consider the set of the first $n-1$ differences between consecutive A_i, that is, $\{d_1 = A_2 - A_1, d_2 = A_3 - A_2, \ldots, d_{n-1} = A_n - A_{n-1}\}$. Let n_0 be the number of differences in this set such that $d_i \geq si^{s-1}f(i)g(i)$, where $g(x)$ is a function of slow increase. Then we have the following limit*

$$\lim_{n\to\infty} \frac{n_0}{n} = 0$$

Therefore if n_1 is the number of differences in this set such that $d_i < si^{s-1}f(i)g(i)$ we have that $\lim_{n\to\infty} \frac{n_1}{n} = 1$. That is, almost all the differences d_i in the set satisfy $d_i = A_{i+1} - A_i < si^{s-1}f(i)g(i)$.

Proof. See [1]. The lemma is proved.

Theorem 2.3 *Let $h(x)$ be a function of slow increase. Let us consider the set of the first $n-1$ differences $\log A_{i+1} - \log A_i$, that is,*

$$\{\log A_2 - \log A_1, \log A_3 - \log A_2, \ldots, \log A_n - \log A_{n-1}\}$$

and let n_3 be the number of differences in this set such that

$$\log A_{i+1} - \log A_i < \frac{\log h(i)}{i} \tag{6}$$

Then

$$\lim_{n\to\infty} \frac{n_3}{n} = 1$$

That is, almost all the differences in the set satisfy inequality (6).

Proof. We have (see (5) and use the formula $(e^x - 1) \sim x$ $(x \to 0)$)

$$A_i\left(e^{\frac{\log h(i)}{i}} - 1\right) \sim i^s f(i)\frac{\log h(i)}{i} = si^{s-1}f(i)\left(\frac{1}{s}\log h(i)\right) \tag{7}$$

where the function $\frac{1}{s}\log h(x)$ is of slow increase (see [2, Theorem 2]). Therefore by Lemma 2.2 and equation (7) the inequality

$$d_i < si^{s-1}f(i)\left(\frac{1}{2}\frac{1}{s}\log h(i)\right) < A_i\left(e^{\frac{\log h(i)}{i}} - 1\right)$$

holds for almost all i $(1 \leq i \leq n-1)$. That is, the inequality

$$1 + \frac{d_i}{A_i} < h(i)^{\frac{1}{i}}$$

holds for almost all i $(1 \leq i \leq n-1)$. That is, the inequality

$$\log A_{i+1} - \log A_i < \frac{\log h(i)}{i}$$

holds for almost all i $(1 \leq i \leq n-1)$. The theorem is proved.

Theorem 2.4 *The inequality*

$$A_n^{\left(\frac{A_{n+1}}{A_n}\right)^n} \leq n^{A_n} \tag{8}$$

holds for almost all n.

Proof. We give two proofs. 1) First proof. The inequality is equivalent to the establishment (see the introduction)

$$\log A_{n+1} - \log A_n \leq \frac{\log A_n}{n} - \frac{\log\log A_n - \log\log n}{n} \tag{9}$$

Theorem 2.3 implies that the following inequality holds for almost all n

$$\log A_{n+1} - \log A_n < \frac{\log h(n)}{n} < \frac{\log n}{n} \tag{10}$$

since (L'Hospital's rule and (4))

$$\lim_{x\to\infty} \frac{\log h(x)}{\log x} = \lim_{x\to\infty} \frac{xh'(x)}{h(x)} = 0$$

Equation (5) gives

$$\log A_n = s\log n + \log f(n) + o(1) = s\log n\left(1 + \frac{\log f(n) + o(1)}{s\log n}\right) \tag{11}$$

That is

$$\frac{\log A_n}{n} = s\frac{\log n}{n} + \frac{\log f(n)}{n} + o\left(\frac{1}{n}\right) \tag{12}$$

Equation (11) gives

$$\log\log A_n = \log s + \log\log n + o(1)$$

That is

$$\frac{\log\log A_n - \log\log n}{n} = \frac{\log s}{n} + o\left(\frac{1}{n}\right) \tag{13}$$

Therefore (see (12) and (13)) the following inequality holds

$$\frac{\log n}{n} < \frac{\log A_n}{n} - \frac{\log\log A_n - \log\log n}{n} \tag{14}$$

Inequality (9) is an immediate consequence of (10) and (14). The theorem is proved.

2) Second proof. The inequality is equivalent to the establishment (see the introduction)

$$d_n = A_{n+1} - A_n \leq A_n \left(\left(\frac{A_n \log n}{\log A_n} \right)^{\frac{1}{n}} - 1 \right) = A_n \left(\exp\left(\frac{\log\left(\frac{A_n \log n}{\log A_n}\right)}{n} \right) - 1 \right)$$

Now,

$$\begin{aligned} & A_n \left(\exp\left(\frac{\log\left(\frac{A_n \log n}{\log A_n}\right)}{n} \right) - 1 \right) \sim A_n \frac{\log\left(\frac{A_n \log n}{\log A_n}\right)}{n} \\ = \; & \frac{A_n}{n} \left(\log A_n + \log\log n - \log\log A_n \right) \sim \frac{A_n}{n} \log A_n \\ \sim \; & s n^{s-1} f(n) \log n \end{aligned} \tag{15}$$

since (see the first proof) $\log A_n \sim s \log n$ and $\log\log A_n \sim \log\log n$.

Now, Lemma 2.2 and equation (15) give for almost all n the inequality

$$d_n < s n^{s-1} f(n) \log^{1/2} n < A_n \left(\left(\frac{A_n \log n}{\log A_n} \right)^{\frac{1}{n}} - 1 \right)$$

The theorem is proved.

Corollary 2.5 *Let p_n be the n-th prime number. The inequality*

$$p_n^{\left(\frac{p_{n+1}}{p_n}\right)^n} \leq n^{p_n}$$

holds for almost all n.

Proof. It is an immediate consequence of Theorem 2.4 when $s = 1$ and $f(n) = \log n$ in equation (5), since (prime number theorem) $p_n \sim n \log n$. The corollary is proved.

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