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Jackson Network as Network of Multiphase Type

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Abstract

Open queuing network with finite number of nodes, infinite number of servers in nodes, exponential distributions of service times and Poisson input flow is considered. A presence of infinite number of servers in the network nodes together with the statement that the cardinality of counting set of counting sets is counting set also allows to transform initial queuing network into queuing network of multiphase type

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so that in each node a customer may be served no more than once. It is proved that all flows of so transformed network in stationary regime are Poisson. Synergetic effects in this network are analysed using a replacement of infinite number of servers by finite number of servers. Possible generalizations of Jackson network permitting suggested analysis are considered.

Mathematics Subject Classification: 60K25, 90B22

Keywords: an acyclic queuing network, a queuing network of multiphase type, a sham node, Poisson flows, a stationary regime, a synergetic effect

Introduction

Open queuing network with finite number of nodes, infinite number of servers in nodes, exponential distributions of service times and Poisson input flow is considered. A presence of infinite number of servers in the network nodes [1] - [3] together with the statement that the cardinality of counting set of counting sets is counting set also allows to transform initial queuing network into queuing network of multiphase type so that in each node a customer may be served no more than once. A transformation of the Jackson network into the multiphase type network is closely connected with models of retrial queues [4], [5].

It is proved that all flows of so transformed network in stationary regime are Poisson. Synergetic effects in this network are analysed using a replacement of infinite number of servers by finite number of them. Synergetic effect means that if number of servers in nodes and intensity of input flow increase in $n \to \infty$ times then probability of queues existence on finite time interval tends to zero.

This investigation is based on the Burke theorem [6] that in stationary regime output flow of multiserver system $M|M|n|\infty$ is Poisson and this system possesses the synergetic effect [7].

1 Transformation of Jackson network into multiphase type network

Consider open queuing network S with finite number of nodes $U = \{0, 1, \dots, m\}$ and Poisson input flow with the intensity λ_0. Paths of customers in the network S are defined by the route matrix $\Theta = ||\theta_{i,j}||_{i,j=0}^m$, consisting of probabilities $\theta_{i,j}$ of customers transitions from the node i to the node j after a service in the node

i. The node 0 is a source of customers arriving the network and a container of customers departing the network. Here $\theta_{0,i}$ is the probability that input flow customer moves to the node i and $\theta_{i,0}$ is the probability that customer departs network after service in the node i. In the node k of the network S there is infinite number of identical servers with service times which has the distribution $F_k(t) = 1 - \exp(-\mu_k t),\ t \geq 0,\ \mu_k,\ 0 < \mu_k < \infty,\ k = 1, \ldots, m,\ \theta_{0,0} = 0$.

Transform the network S into the following network S^*. Each node $k,\ 0 \leq k \leq m$, is divided into infinite number of nodes $(k, j),\ 1 \leq j$. Here nodes with $1 \leq k \leq m$ are nodes with infinite numbers of servers and nodes with $k = 0$ absorb customers departing the network. A customer arriving the network with the probability $\theta_{0,k}$ moves to the node $(k, 1)$. The node $(0, 1)$ is sham because $\theta_{0.0} = 0$ and so customers do not visit it. Then after a service in the node $(p, j),\ 1 \leq p \leq m,\ 1 \leq j$, customer with the probability $\theta_{p,q}$ moves to the node $(q, j+1)$ and with the probability $\theta_{p,0}$ moves to the node $(0, j+1)$ - departs the network, $1 \leq p, q \leq m,\ 1 \leq j$. Consequently initial network S is transformed into the network S^* with the nodes set $U^* = \{(k, j),\ 1 \leq j,\ 0 \leq k \leq m\}$. Graphically the network S^* is represented in Fig. 1.

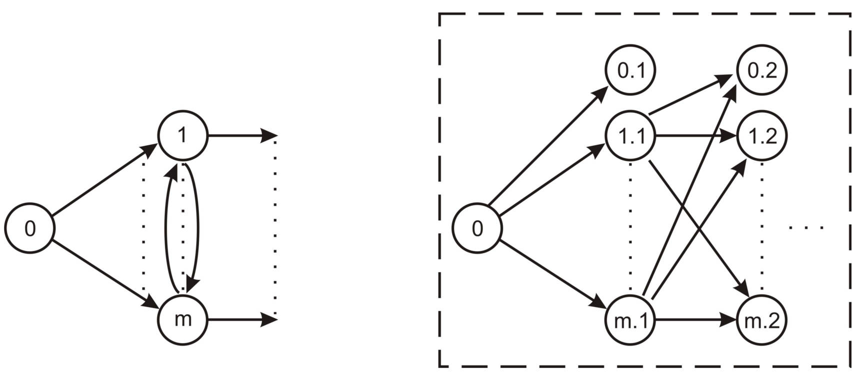

Fig. 1. Transformation of Jackson network (leftward) into multiphase type network (rightward).

The network S^* is constructed similar to retrial queues systems [1] - [5]. Transformation of the network S into the network S^* does not change paths and service times of customers.

In the network S^* a system of balance equations for stationary intensities of flows arriving the nodes of the set U^* may be solved by recurrent relations

$$\lambda_{k,1} = \lambda_0 \theta_{0,k},\ \lambda_{k,j+1} = \sum_{p=1}^{m} \lambda_{p,j} \theta_{p,k},\ 0 \leq k \leq m,\ 1 \leq j. \tag{1}$$

Assume that the condition $\min\limits_{1\leq p\leq m}\theta_{p,0}=\underline{\theta}>0$ is true then the following inequalities take place

$$\sum_{k=1}^{m}\lambda_{k,j+1}\leq(1-\underline{\theta})\sum_{p=1}^{m}\lambda_{p,j}\leq(1-\underline{\theta})^{j}\lambda_{0},\ 0\leq k\leq m, \tag{2}$$

consequently the series $\lambda_k=\sum\limits_{j\geq 1}\lambda_{k,j},\ 0\leq k\leq m$, converge and from the equalities (1) we have

$$\lambda_k=\sum_{p=0}^{m}\lambda_p\theta_{p,k},\ 1\leq k\leq m. \tag{3}$$

So vector-line $(\lambda_1,\dots,\lambda_m)$ satisfies the system (3) of balance equations for open network S with fixed λ_0. From the condition $\underline{\theta}_0>0$ and theorem of Frobenius-Perron [8, Chapter XIII] this solution is single.

Indeed, denote $\Theta_0=||\theta_{i,j}||_{i,j=1}^{m}$. Then system (3) may be rewritten as follows

$$(\lambda_1,\dots,\lambda_m)=(\lambda_1,\dots,\lambda_m)\Theta_0+\lambda_0(\theta_{0,1},\dots,\theta_{0,m}).$$

General solution of this nonuniform system of linear equations equals a sum of partial solution

$$\lambda_0(\theta_{0,1},\dots,\theta_{0,m})\sum_{k\geq 0}\Theta_0^k \tag{4}$$

of nonuniform system (3) and general solution of the uniform system

$$(\lambda'_1,\dots,\lambda'_m)=(\lambda'_1,\dots,\lambda'_m)\Theta_0.$$

From theorem of Frobenius-Perron and the condition $\underline{\theta}>0$ modules of eigen values of the matrix Θ_0 are smaller one. So general solution of the uniform system equals zero. Consequently nonuniform system (3) has single solution represented in Formula (4).

2 Multiphase type queuing networks with finite number of phases

Consider the network S_N^*, with the nodes set $U_N^*=\{(k,j),\ 0\leq k\leq m,\ 1\leq j\leq N\}$, in which all customers departing the nodes $(k,N),\ 1\leq k\leq m$, leave the network. So in the network S_N^*, in a contradistinction to the network S^* each customer may be served no more than N times. It is obvious that input flow to the node (k,j) of the network S_N^* coincides with input flow to the node (k,j) of the network S^*.

From (1) we have that for fixed $\lambda_0 > 0$ there is single solution of balance equations system for the network S_N^*. As each node of the network S_N^* contains infinite number of servers so process describing numbers of customers in this network nodes is ergodic. Consequently product Jackson theorem [9] asserting that limit distribution of customers numbers in the network S_N^* nodes equals product of limit distributions of customers numbers in isolated nodes provided that input flows to isolated nodes are Poisson with intensities defined by systems of balance equations (1) is true.

Assume that in initial time moment a distribution of customers numbers in the network S_N^* is stationary. As the network S_N^* has multiphase type then from [6], [10], [11, Corollary 3.2] we have that all flows departing the nodes of the network S_N^* are Poisson.

Remark 2.1 *For sufficiently large N the network S_N^* approximate the network S^* in the following sense. Each customer reaches the nodes (k,N), $1 \leq k \leq m$, with probability smaller than $(1-\underline{\theta})^{N-1} \to 0$, $N \to \infty$. Note that for $j \leq N$ input flow to the node (k,j) of the network S_N^* coincides with input flows to the node (k,j) of the networks S_{N+1}^*, $S_{N+2}^*, \dots,$ S^*.*

3 Synergetic effects in multiphase type networks

Fix ε, $0 < \varepsilon < 1$ and assume that positive numbers $\alpha_{k,j}$ satisfy the equalities

$$\alpha_{k,j} = \frac{\lambda_{k,j}}{\mu_k(1-\varepsilon)^2},\ (k,j) \in U_N^*. \tag{5}$$

Define integers $n_{k,j}$ by the equalities $n_{k,j} = [n\alpha_{k,j}]$, $(k,j) \in U_N^*$, where $[a]$ is integer part of real number a. Transform the network S_N^* into the network $S_{N,n}^*$, in which the node $(k,j) \in U_N^*$ contains $n_{k,j}$ identical servers with exponential distribution of service times and parameter μ_k, and input flow is Poisson with the intensity $n\lambda_0$.

From the equalities (5) it is not difficult to obtain the inequalities

$$\frac{n\lambda_{k,j}}{n_{k,j}\mu_k} < 1-\varepsilon,\ (k,j) \in U_N^*,\ \text{for}\ n > \max_{(k,j)\in U_N^*} \frac{1}{\varepsilon a_{k,j}}. \tag{6}$$

From Formula (6) we have that process describing numbers of customers in nodes of the network $S_{N,n}^*$ is ergodic. Assuming that initial distribution of customers numbers in the network $S_{N,n}^*$ nodes is stationary we obtain that input flows to nodes (k,j) of this network are Poisson with the intensities $\lambda_{k,j}$, $(k,j) \in U_N^*$.

From [12] we have that limit probability $P_n(k,j)$ of customers absence in the node (k,j) of the network $S^*_{N,n}$ from the condition (6) satisfies the relation

$$P_n(k,j) \to 1,\ n \to \infty,\ (k,j) \in U^*_N. \tag{7}$$

Fix time segment $[0,T]$ and denote $P^T_n(k,j)$ the probability of queues absence in the node (k,j) on the segment $[0,T]$. Using results of [7, § 3] and the relation (7) it is simple to obtain limit relation

$$P^T_n(k,j) \to 1,\ n \to \infty,\ (k,j) \in U^*_N. \tag{8}$$

Denote $P^T_n(N)$ the probability of queues absence in all nodes of the network $S^*_{N,n}$. Using Formula (8) we obtain limit relation

$$P^T_n(N) \to 1,\ n \to \infty. \tag{9}$$

4 Generalizations of multiphase type queuing networks

There are following possible generalizations of Jackson networks with infinite numbers of servers in their nodes. All these generalizations possess synergetic effects (9).

1) Jackson network with few types of customers,

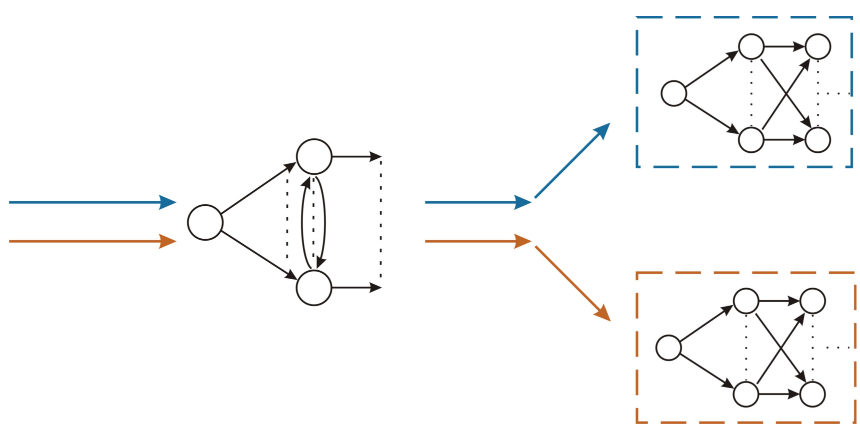

Fig. 2. Network with few types of customers flows (marked by different colours).

2) Jackson network with hyperexponential distributions of customers service times,

Fig. 3. Transformation of system with hyperexponential service times distribution into network with exponential times distributions.

3) Jackson network with dependence of probability characteristics of paths and service times on numbers of customers services.

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