On the Numerical Range of a Generalized Derivation

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Abstract
We examine the relationship between the numerical range of the restriction of a generalized derivation to a norm ideal J and that of its implementing elements.

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1 Introduction

Given a Banach algebra $\mathcal{A}$, $\mathcal{A}^*$ the dual of $\mathcal{A}$, $S(\mathcal{A}) = \{x \in \mathcal{A} : \|x\| = 1\}$, the unit sphere, and $x \in S(\mathcal{A})$, let $D(x, \mathcal{A}) = \{f \in \mathcal{A}^* : f(x) = 1 = \|f\|\}$. 
The Hahn-Banach theorem guarantees that $D(x, \mathcal{A})$ is non empty for each $x \in S(\mathcal{A})$. The elements of $D(I, \mathcal{A})$, $I$, the identity in $\mathcal{A}$, are called normalized states or simply states.

For $a \in \mathcal{A}$, and $x \in S(\mathcal{A})$, we define $V(x, a, \mathcal{A}) = \{f(ax) : f \in D(x, \mathcal{A})\}$. The numerical range of $a$ is the set $V(a, \mathcal{A}) = \bigcup \{V(x, a, \mathcal{A}) : x \in S(\mathcal{A})\}$. Given a Banach space $\mathcal{H}$, we may consider the Banach algebra $\mathcal{A} = L(\mathcal{H})$ and define $\mathcal{D}$ the spatial numerical range of $A$ by

$$W(A; L(\mathcal{H})) = \{f(Ax) : f \in \mathcal{H}^*, x \in \mathcal{H}, \text{and } \|f\| = \|x\| = 1 = f(x)\}$$

We first give some basic properties of the numerical range.

Bonsal [4], has shown that $V(a, \mathcal{A}) = V(I, a, \mathcal{A})$, and for each $a \in \mathcal{A}$, $V(a, \mathcal{A})$ is a compact convex subset of $\mathbb{C}$.

**Lemma 1.** $V(x, a, \mathcal{A}) = \{f(ax) : f \in D(x, \mathcal{A})\}$ is convex.

**Proof.** Let $\lambda_1, \lambda_2 \in V(x, a, \mathcal{A})$. Then there exist support functionals $f_1, f_2 \in D(x, \mathcal{A})$ such that $\lambda_1 = f_1(ax), \lambda_2 = f_2(ax)$.

Define $f$ on $D(x, \mathcal{A})$ by $f(ax) = tf_1(ax) + (1-t)f_2(ax), t \in (0, 1)$. We need to show that $f \in D(I, \mathcal{A})$. Clearly $f$ is linear and $|f(ax)| = |tf_1(ax) + (1-t)f_2(ax)| \leq t|f_1(ax)| + (1-t)|f_2(ax)| \leq t\|f_1\|\|ax\| + (1-t)\|f_2\|\|ax\| = \|ax\| \Rightarrow \|f\| \leq 1$.

Also, $f(x) = tf_1(x) + (1-t)f_2(x) = 1$ \Rightarrow $\|f\| \geq 1$

Thus $f \in D(I, \mathcal{A})$ which is convex and hence $V(x, a, \mathcal{A})$ is convex. \qed

For $a \in \mathcal{A}$, we define the left multiplication operator $L_a : \mathcal{A} \to \mathcal{A}$ by $L_a(x) = ax, \forall x \in \mathcal{A}$ and $\|L_a\| = \sup \{\|ax\| : x \in \mathcal{A}, \|x\| \leq 1\}$

$L_a$ is a linear operator in $\mathcal{A}$ and also a bounded operator since $\|L_a\| = \sup \{\|ax\| : x \in \mathcal{A}, \|x\| \leq 1\} \leq \|a\|$.

$L_a(\mathcal{A})$ will denote the set of all left multiplication operators on the algebra $\mathcal{A}$ as $a$ ranges on $\mathcal{A}$. This set is a normed algebra.

The algebraic numerical range of $L_a \in L_a(\mathcal{A})$ is the non-empty set:

$V(L_a; L_a(\mathcal{A})) = \{f(L_a) : f \in L_a(\mathcal{A})^*, f(L_e) = 1 = \|f\|\}$.

Similarly the right multiplication operator for $b \in \mathcal{A}$ is defined as:

$R_b : \mathcal{A} \to \mathcal{A}, x \mapsto xb$

We note that $\forall x \in \mathcal{A}$ and fixed $a, b \in \mathcal{A}$, $\Delta_{a,b}(x) = L_a(x) - R_b(x) = ax - xb$, is the generalized derivation induced by $a, b \in \mathcal{A}$.

In [3], it is shown that for any Banach algebra $\mathcal{A}$, $\|L_a\| = \|a\| = \|R_a\|$ and that $V(a; \mathcal{A}) = V(L_a; L(\mathcal{A})) = V(R_a; L(\mathcal{A}))$, $L(\mathcal{A})$ the algebra of the bounded linear operators on $\mathcal{A}$.

**Lemma 2.** For $a \in \mathcal{A}$, $L_a \in L_a(\mathcal{A}), \|L_a\| = \|a\| = \|R_a\|$
Proof.

\[ \|L_a\| = \sup \{ \|L_a(x)\| : \|x\| = 1 \} \]
\[ = \sup \{ \|ax\| : \|x\| = 1 \} \]
\[ \leq \|a\| \|x\| \]
\[ \Rightarrow \|L_a\| \leq \|a\| \text{...............}(i) \]

If \( \mathcal{A} \) has unit \( e \), we have \( L_a(e) = ae = a \) which implies \( \|a\| = \|L_a(e)\| \leq \|L_a\| \|e\| = \|L_a\| \Rightarrow \|L_a\| \geq \|a\| \text{...............}(ii) \)

From (i) and (ii) equality follows.

Similarly we obtain \( \|R_a\| = \|a\| \).

Lemma 3. For \( a \in \mathcal{A}, V(a; \mathcal{A}) = V(L_a; L(\mathcal{A})) = V(R_a; L(\mathcal{A})) \)

Proof. Let \( \lambda \in V(a; \mathcal{A}) \), Then there exist \( f \in S(\mathcal{A}) \) such that \( f(a) = \lambda \)

Now define \( F \) on \( L(\mathcal{A}) \) by

\[ F(L_a) = f(ax), \text{ for all } L_a \in L(\mathcal{A}). \]

Clearly \( F \) is linear since

\[ F(\alpha L_a + \beta L_b) = f(\alpha ax + \beta bx) \]
\[ = f(\alpha ax) + f(\beta bx) \]
\[ = \alpha f(ax) + \beta f(bx) \]
\[ = \alpha F(L_a) + \beta F(L_b), a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C} \]

\( f \) is also bounded and positive since

\[ \|F(L_a)\| = \sup \{ \|f(ax)\| \} \leq \|f\| \|ax\| = c \|L_a\|. \]

Also \( F(L_a) = f(e) = f(x) = 1 \) and \( \|F\| = 1 \).

So \( F \) as defined is a positive linear functional on \( \mathcal{A} \).

Take a finite rank operator \( b \in L(\mathcal{A}) \) defined by

\( bx = g(x)a \), for all \( x \in \mathcal{A}, g \in S(\mathcal{A}) \). Clearly \( \|b\| = 1 \) and \( F(b) = f(bx) = f(g(x)a) = g(x)f(a) = \lambda \). Hence \( V(a; \mathcal{A}) \subseteq V(L_a; L(\mathcal{A})) \)

Conversely we show that \( V(L_a; L(\mathcal{A})) \subseteq V(a; \mathcal{A}) \)

Let \( \lambda \in V(L_a; L(\mathcal{A})) \). Then there exists a state \( f \in L(\mathcal{A})^* \) such that \( f(L_a) = \lambda \)

Define a functional \( h \in \mathcal{A}^* \) by \( h(a) = f(L_a) \). Then :

\[ h(\alpha a + \beta b) = f(\alpha L_a + \beta L_b) \]
\[ = f(\alpha L_a) + f(\beta L_b) \]
\[ = \alpha f(L_a) + \beta f(L_b) \]
\[ = \alpha h(a) + \beta h(b) \]
$\Rightarrow h$ is linear and bounded. $h$ is also positive since $h(a^*a) = f(L_a^*L_a) \geq 0$
Furthermore $h$ is of norm 1 since $h(e) = f(L_e) = 1$ and
$1 = |h(e)| \leq \|h\|\|e\| \Rightarrow \|h\| \geq 1$. We also have
\[
\|h\| = \sup \{|h(a)| : \|a\| = 1\} \\
= \sup \{|f(L_a)| : \|L_a\| = 1\} \\
\leq \|f\| \\
= 1
\]
Thus $h$ is a state on $\mathcal{A}^*$ and so $V(L_a; L(\mathcal{A})) \subseteq V(a; \mathcal{A})$

\section{NORM IDEALS}

Let $X$ and $Y$ be Banach algebras. $L(X)$ and $L(Y)$, the algebra of all bounded linear operators on $X$ and $Y$ respectively.
Let $(J, \|\|_J)$ be a norm ideal on $L(Y, X)$, the algebra of all bounded linear operator from Y to X such that:

i) $(J, \|\|_J)$ is a Banach space

ii) If $A \in L(X), T \in J, B \in L(Y)$ then $ATB \in J$, and $\|ATB\|_J \leq \|A\| \|T\|_J \|B\|

iii) $\|T\| \leq \|T\|_J$, $T \in J$ and

iv) $\|T\|_J = \|T\|$, for $T$ a rank- one operator.

If $A \in L(X), B \in L(Y)$ and $T \in J$, then the operators $L_A, R_B$ and $L_A - R_B$ are all bounded linear operators on $L(J)$, the space of all bounded linear operators from $J$ to $J$, where:
$L_A T = AT$, the left multiplication operator,
$R_B T = TB$, the right multiplication operator and
$(L_A - R_B) T = AT - TB$, the generalized derivation. The following lemma will hold.

\textbf{Lemma 4.} $V(A : L(X)) = V(L_A : L(J))$

\textit{Proof.} Let $\lambda \in V(A : L(X))$. Then there exist $f \in L(X)^*$ such that
$\lambda = f(A)$, and $f(I_{L(X)}) = 1 = \|f\|
Let $\mathcal{A}_0 = \{L_A : A \in L(X), L_A(T) = AT, T \in J\} \subseteq L(J)$.
$\mathcal{A}_0$ is a linear subspace of $L(X)$.

On $\mathcal{A}_0^*$, define a linear functional $g$ such that $g(L_A) = f(A)$. Clearly $g$ as defined is a state and the Hahn-Banach theorem guarantees the existence of its extension on $L(J)$. Hence, $V(A : L(X)) \subseteq V(L_A : L(J))$
$\Leftrightarrow$ suppose $\lambda \in V(L_A : L(J))$. Then $\exists f \in L(J)^*$ such that $f(L_A) = \lambda$ and
Define a linear operator $h$ on $L(X)^*$ by $h(A) = f(L_A)$. Then $h(I) = f(I_{L(J)}) = 1$.

$h$ is thus a state on $L(X)^*$ and $V(L_A : L(J)) \subseteq V(A : L(X))$.

\section{Norm of $L_A$ and $R_B$ in $(J, \| \cdot \|_J)$}

\begin{lemma}
$\|L_A\|_J = \|A\|$.
\end{lemma}

\begin{proof}
Condition (ii) above on the definition of a norm ideal implies that $L_A$ and $R_B$ are bounded linear operators on $(J, \| \cdot \|_J)$ and

$$
\|L_A\|_J = \text{Sup} \{\|AX\| : \|X\|_J = 1, X \in J\} \\
\leq \|A\| \|X\|_J \\
= \|A\|
$$

Condition (iii) implies $\|L_A\|_J \geq \|A\|$.

It therefore follows that $\|L_A\|_J = \|A\|$.

Similarly $\|R_B\|_J = \|B\|$.
\end{proof}

\section{Numerical range of the generalized derivation in the norm ideal $J$}

In the past, generalized derivations, their properties and their restrictions to norm ideals have been investigated by many authors. For example, their spectra have been characterized in [7] and [8]. The famous results on the norms of inner derivation and the generalized derivation as obtained by Stampfli [6] using maximal numerical range have ever since provided a crucial lead in defining of norms of elementary operators. We recall the works of Kyle [9] who examines the relationship between the numerical range of an inner derivation, and that of its implementing element.

In his paper, Magajna [2] gives the essential numerical range of the the generalized derivation defined on the Hilbert-Schmidt class in terms of the numerical and the essential numerical ranges of the implementing operators. Shaw [10] in particular, established that the algebra numerical range of a generalized derivation restricted to a norm ideal $J$ is equal to the difference of the algebra numerical ranges of the implementing operators provided that $J$ contains all finite rank operators and is suitably normed. With slight modification we obtain an alternative proof to Shaw’s result.

\begin{lemma}
Let $J$ be as defined above. Then for $A \in L(X), B \in L(Y), V(\Delta_{A,B} : L(J)) = V(A : L(X)) - V(B : L(Y))$.
\end{lemma}
Proof. Let \( \lambda \in V(\Delta_{A,B} : L(J)) \). This implies there exist \( f \in L(J)^* \) such that
\[
\lambda = f(\Delta_{A,B} : L(J)) = \{ f(L_A - R_B : f \in S(L(J))) \}
\]
\[
= \{ f(L_A) : f \in L(X)^*, f(I_{L(X)}) = 1 = \|f\| \}
\]
\[
- \{ f(R_B) : f \in L(Y)^*, f(I_{L(Y)}) = 1 = \|f\| \}
\]
\[
= V(L_A : L_A \in L(J)) - V(R_B : R_B \in L(J))
\]
\[
\subseteq V(A : L(X)) - V(B : L(Y))
\]
To prove the reverse inclusion, we make use of the spatial numerical range. Choose \( \lambda \) in \( W(A : L(X)) \) and \( \mu \) in \( W(B : L(Y)) \). Then we can find functionals \( f \) and \( g \) in \( L(X)^*, L(Y)^* \) such that
\[
\|f\| = \|x\| = f(x) = 1, \text{ with } f(Ax) = \lambda \text{ and}
\|g\| = \|y\| = g(y) = 1, \text{ with } g(By) = \mu
\]
Let \( X \) be a rank one operator in \( J \) such that \( Xz = g(z)x, \forall z \in Y \),
Also define \( F \) in \( L(J)^* \) by \( F(T) = f(Ty), \forall T \in L(J) \)
Then \( F(X) = f(Xy) = fg(y)x = g(y)f(x) = 1 \),
\( F(I) = f(Iy) = fg(y)x = g(y)f(x) = 1 \) and
\( |F(T)| \leq \|f\| \|T\|_J \|Y\| = \|T\|_J \)
Clearly \( \|F\|_J = \|X\|_J = 1 \) and \( (I_{L(J)}, F) \in L(J) \times L(J)^* \)
Thus,
\[
F(\Delta_{A,B}(X)) = F(AX - XB)
\]
\[
= f(AX - XB)y
\]
\[
= f(AXy) - f(XBy)
\]
\[
= f(g(y)Ax) - f(g(By)x)
\]
\[
= f(Ax)g(y) - f(x)g(By)
\]
\[
= \lambda - \mu
\]
\[
\in \{ W(A : L(X)) - W(B : L(Y)) \}
\]
Now
\[
V(\Delta_{A,B} : L(J)) = \sigma W(\Delta_{A,B} : L(J))
\]
\[
\geq \sigma \{ W(A ; L(X)) - W(B ; L(Y)) \}
\]
\[
= \sigma \{ W(A ; L(X)) \} - \sigma \{ W(B ; L(Y)) \}
\]
\[
= V(A ; L(X)) - V(B ; L(Y))
\]
Thus \( \{ V(A; L(X)) - V(B; L(Y)) \} \subseteq V(\Delta_{A,B}; L(J)) \)

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**References**


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