An Isomorphism Theorem for Bornological Groups

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Abstract

A necessary and sufficient condition for the validity of a general form of the isomorphism theorem in the context of bornological groups is established.

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1. Introduction

The notion of a bornological group has been defined and studied in [6]: a bornological group is a pair \((G, B)\) consisting of a group \(G\) and a bornology \(B\) on \(G\) in such a way that the mapping

\[(x, y) \in (G \times G, B \times B) \mapsto xy^{-1} \in (G, B)\]

is bounded, where \(B \times B\) is the product bornology on \(G \times G\). In that paper a version of the isomorphism theorem for bornological groups has been obtained, as well as some consequences of it. The purpose of this note is to establish a more general version of the isomorphism theorem in the context of bornological groups.
2. The result

**Theorem 1.** Let \((G, \mathcal{B})\) and \((H, \mathcal{C})\) be two bornological groups, \(u: (G, \mathcal{B}) \to (H, \mathcal{C})\) a surjective bounded group homomorphism, \(N' = u^{-1}(N')\) (which is a normal subgroup of \(H\) and \(N = u^{-1}(N)\)). Let \(\pi: G \to G/N\) and \(\pi': H \to H/N'\) be the canonical surjections and consider \(G/N\) and \(H/N'\) endowed with the corresponding quotient bornologies \(\mathcal{B}' = \pi(\mathcal{B})\) and \(\mathcal{C}' = \pi'(\mathcal{C})\) (which are group bornologies). In order that there exists a (necessarily unique) bornological group isomorphism \(\tilde{u}: (G/N, \mathcal{B}') \to (H/N', \mathcal{C}')\) making the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{u} & H \\
\pi \downarrow & & \downarrow \pi' \\
G/N & \xrightarrow{\tilde{u}} & H/N'
\end{array}
\]

commutative, it is necessary and sufficient that for each \(C \in \mathcal{C}\) there exists a \(B \in \mathcal{B}\) such that \(u^{-1}(C) \subset BN\).

**Proof.** First, let us prove the necessity. Indeed, let \(\tilde{u}: (G/N, \mathcal{B}') \to (H/N', \mathcal{C}')\) be a bornological group isomorphism as in the statement of the theorem. Put \(w = \pi' \circ u\), which is a bounded group homomorphism from \((G, \mathcal{B})\) onto \((H/N', \mathcal{C}')\) such that \(\text{Ker}(w) = N\) and \(\tilde{u} \circ \pi = w\). Therefore, by the necessity of Theorem 3.16 of [6], for each \(C \in \mathcal{C}\) there is a \(B \in \mathcal{B}\) so that \(u^{-1}(C) \subset BN\). Consequently, \(w^{-1}(C) \subset BN\). In fact, if \(x \in u^{-1}(C)\), \(w(x) = \pi'(u(x)) \in \pi'(C)\) and hence there are \(b \in B\) and \(n \in N\) so that \(x = bn\).

Now, let us prove the sufficiency. Indeed, let \(w\) be as above and let \(C \in \mathcal{C}\) be arbitrary. By hypothesis, there is a \(B \in \mathcal{B}\) so that \(u^{-1}(C) \subset BN\). We claim that

\(w^{-1}(\pi'(C)) \subset BN\).

In fact, if \(x \in w^{-1}(\pi'(C))\),

\(u(x)N' \in \pi'(C) = \pi'(u(u^{-1}(C))) \subset \pi'(u(BN)) = \pi'(u(B))\pi'(N') = \pi'(u(B))\),

and so there is a \(b \in B\) so that \(u(x)N' = u(b)N'\). Thus \(u(xb^{-1}) \in N'\), and \(xb^{-1} \in N\), that is, \(x \in BN\). Therefore, by the sufficiency of Theorem 3.16 of [6], there exists a (necessarily unique) bornological group isomorphism \(\tilde{u}: (G/N, \mathcal{B}') \to (H/N', \mathcal{C}')\) making the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{w} & H/N' \\
\pi \downarrow & & \downarrow \pi' \\
G/N & \xrightarrow{\tilde{u}} & H/N'
\end{array}
\]

commutative. This completes the proof of the theorem.
We have obtained Theorem 1 from Theorem 3.16 of [6], but we could have given a direct proof of Theorem 1, following the lines of that of Theorem 3.16 of [6]. On the other hand, Theorem 3.16 of [6] is an immediate consequence of Theorem 1:

**Corollary 2.** Let \((G, \mathcal{B}), (H, \mathcal{C})\) and \(u\) be as in Theorem 1. Let \(\pi: G \to G/\text{Ker}(u)\) be the canonical surjection and let \(\mathcal{B}'\) be the quotient bornology on \(G/\text{Ker}(u)\). In order that there exists a (necessarily unique) bornological group isomorphism \(\tilde{u}: (G/\text{Ker}(u), \mathcal{B}') \to (H, \mathcal{C})\) making the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{u} & H \\
\pi \downarrow & & \downarrow \tilde{u} \\
G/\text{Ker}(u) & & \\
\end{array}
\]

commutative, it is necessary and sufficient that for each \(C \in \mathcal{C}\) there exists a \(B \in \mathcal{B}\) such that \(u^{-1}(C) \subset B \text{Ker}(u)\).

**Proof.** Follows immediately from Theorem 1, by taking \(N'\) as the subgroup of \(H\) reduced to its identity element.

**Remark 3.** Theorem 1 implies a well-known isomorphism theorem for groups ([3], p.17, (v)), which reads:

Let \(G\) and \(H\) be two groups, \(u: G \to H\) a group homomorphism, \(N'\) a normal subgroup of \(H\) and \(N = u^{-1}(N')\). Then there exists a unique injective group homomorphism \(\tilde{u}: G/N \to H/N'\) making the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{u} & H \\
\pi \downarrow & & \downarrow \pi' \\
G/N & \xrightarrow{\tilde{u}} & H/N' \\
\end{array}
\]

commutative.

Indeed, by a straightforward argument, we may assume that \(u\) is surjective. Consider \(G\) (resp. \(H\)) endowed with the group bornology \(\mathcal{B}\) (resp. \(\mathcal{C}\)) consisting of all subsets of \(G\) (resp. \(H\)). Then \(u: (G, \mathcal{B}) \to (H, \mathcal{C})\) is a bounded group homomorphism. Moreover, for all \(C \in \mathcal{C}\), \(u^{-1}(C) \in \mathcal{B}\) and \(u^{-1}(C) \subset u^{-1}(C)N\). Thus our claim is an immediate consequence of Theorem 1, \(\tilde{u}\) being a group isomorphism from \(G/N\) onto \(H/N'\) in this case.

**Remark 4.** Theorem 1 implies the corresponding results for bornological modules [5], linearly bornologized modules [4], bornological vector spaces [7], convex bornological vector spaces over \(\mathbb{R}\) or \(\mathbb{C}\) [2], and convex bornological vector spaces over a complete non-Archimedean non-trivially valued field [1].
For instance, we shall prove the version of Theorem 1 for bornological \( R \)-modules, \( R \) being a topological ring with a non-zero identity element. Let us first observe that if \((E, \mathcal{B})\) is a bornological \( R \)-module, then \((E, +, \mathcal{B})\) is an abelian bornological group; moreover, if \((F, \mathcal{C})\) is a bornological \( R \)-module and \(u: (E, \mathcal{B}) \to (F, \mathcal{C})\) is a bounded \( R \)-linear mapping, then \(u: ((E, +), \mathcal{B}) \to ((F, +), \mathcal{C})\) is a bounded group homomorphism.

We shall show the validity of the following result:

Let \((E, \mathcal{B})\) and \((F, \mathcal{C})\) be two bornological \( R \)-modules, \(u: (E, \mathcal{B}) \to (F, \mathcal{C})\) a surjective bounded \( R \)-linear mapping, \(M'\) a submodule of \(F\) and \(M = u^{-1}(M')\). Let \(\pi: E \to E/M\) and \(\pi': F \to F/M'\) be the canonical surjections, and consider \(E/M\) and \(F/M'\) endowed with the corresponding quotient bornologies \(\mathcal{B}'\) and \(\mathcal{C}'\) (which are \( R \)-module bornologies). In order that there exists a (necessarily unique) bornological \( R \)-module isomorphism \(\tilde{u}: (E/M, \mathcal{B}') \to (F/M', \mathcal{C}')\) making the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{u} & F \\
\downarrow{\pi} & & \downarrow{\pi'} \\
E/M & \xrightarrow{\tilde{u}} & F/M'
\end{array}
\]

commutative, it is necessary and sufficient that for each \(C \in \mathcal{C}\) there exists a \(B \in \mathcal{B}\) such that \(u^{-1}(C) \subset B + M\).

The existence of a bornological \( R \)-module isomorphism \(\tilde{u}\) as above and the necessity of Theorem 1 imply that for each \(C \in \mathcal{C}\) there is a \(B \in \mathcal{B}\) such that \(u^{-1}(C) \subset B + M\). Conversely, if the latter condition holds, the sufficiency of Theorem 1 guarantees the existence of a (necessarily unique) bornological group isomorphism \(\tilde{u}: ((E/M, +), \mathcal{B}') \to ((F/M', +), \mathcal{C}')\) such that \(\tilde{u} \circ \pi = \pi' \circ u\). Finally, \(\tilde{u}\) is a bornological \( R \)-module isomorphism, since \(\tilde{u}\) is an \( R \)-linear mapping in the present case.

References


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