A Note on ∗-Derivations of Prime ∗-Rings

Kyung Ho Kim
Department of Mathematics
Korea National University of Transportation
Chungju 380-702, Korea

Yong Hoon Lee
Department of Mathematics, Dankook University
Cheonan 330-714, Korea

Abstract
The aim of the present paper is to establish some results involving ∗-derivations in ∗-rings and investigate the commutativity of prime ∗-rings admitting ∗-derivations of $R$ satisfying certain identities and some related results have also been discussed.

Mathematics Subject Classification: Primary 16Y30

Keywords: ∗-ring, ∗-derivation, prime, 2-torsion free, commutative

1 Introduction
Over the last few decades, several authors have investigated the relationship between the commutativity of the ring $R$ and certain specific types of derivations of $R$. The first result in this direction is due to E. C. Posner [8] who proved that if a ring $R$ admits a nonzero derivation $d$ such that $[d(x), x] \in Z(R)$ for all $x \in R$, then $R$ is commutative. This result was subsequently, refined and extended by a number of authors. In [7], Bresar and Vuckman showed that a prime ring must be commutative if it admits a nonzero left derivation. Recently, many authors have obtained commutativity theorems for prime and
semiprime rings admitting derivation, generalized derivation. Furthermore, Bresar and Vukman [5] studied the notions of a \( \ast \)-derivation and a Jordan \( \ast \)-derivation of \( R \). The aim of the present paper is to establish some results involving \( \ast \)-derivations in \( \ast \)-rings and investigate the commutativity of prime \( \ast \)-rings admitting \( \ast \)-derivations of \( R \) satisfying certain identities and some related results have also been discussed.

2 Preliminaries

Throughout \( R \) will represent an associative ring with center \( Z(R) \). For all \( x, y \in R \), as a usual commutator, we shall write \( [x, y] = xy - yx \), and \( x \circ y = xy + yx \). Also, we make use of the following two basic identities without any specific mention:

\[
\begin{align*}
    x \circ (yz) &= (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z \\
    (xy) \circ z &= x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z] \\
    [xy, z] &= x[y, z] + [x, z]y, \quad [x, yz] = y[x, z] + [x, y]z.
\end{align*}
\]

Let \( R \) is a ring. Then \( R \) is prime if \( aRb = \{0\} \) implies \( a = 0 \) or \( b = 0 \). An additive mapping \( d : R \to R \) is called a derivation if \( d(xy) = d(x)y + xd(y) \) holds for all \( x, y \in R \). An additive mapping \( x \to x^\ast \) of \( R \) into itself is called an involution if the following conditions are satisfied (i) \( (xy)^\ast = y^\ast x^\ast \) (ii) \( (x^\ast)^\ast = x \) for all \( x, y \in R \). A ring equipped with an involution is called an \( \ast \)-ring or ring with involution. Let \( R \) be a \( \ast \)-ring. An additive mapping \( d : R \to R \) is called an \( \ast \)-derivation if \( d(xy) = d(x)y^\ast + xd(y) \) holds for all \( x, y \in R \). An additive mapping \( d : R \to R \) is called a reverse \( \ast \)-derivation if \( d(xy) = d(y)x^\ast + xd(x) \) holds for all \( x, y \in R \). An additive mapping \( F : R \to R \) is called a generalized derivation if there exists a derivation \( d \) such that \( F(xy) = F(x)y + xd(y) \) for all \( x, y \in R \). Let \( R \) be an \( \ast \)-ring. An additive mapping \( F : R \to R \) is called a generalized \( \ast \)-derivation if there exists an \( \ast \)-derivation such that \( F(xy) = F(x)y^\ast + xd(y) \) for all \( x, y \in R \).

3 \( \ast \)-derivations of prime \( \ast \)-rings

**Theorem 3.1** Let \( R \) be a semiprime \( \ast \)-ring. If \( R \) admits an \( \ast \)-derivation \( d \) of \( R \), then \( d \) maps from \( R \) to \( Z(R) \).

**Proof.** By hypothesis, we have

\[
d(xy) = d(x)y^\ast + xd(y), \quad \forall \ x, y \in R,
\]
Replacing $y$ by $yz$ in (1), we have
\[ d(xyz) = d(x)z^*y^* + xd(y)z^* + xyd(z), \quad \forall \ x, y, z \in R. \quad (2) \]

On the other hand,
\[ d(xyz) = d(xy(z)) = d(x)y^*z^* + xd(y)z^* + xyd(z), \quad \forall \ x, y, z \in R. \quad (3) \]

Combining (2) with (3), we have $d(x)[y^*, z^*] = 0$ for all $x, y, z \in R$. Substituting $y^*$ for $y$ and $z^*$ for $z$ in this relation, we have $d(x)[y, z] = 0$ for all $x, y, z \in R$. Taking $yd(x)$ instead of $y$ in the last relation, we have
\[ d(x)y[d(x), z] = 0, \quad \forall \ x, y, z \in R. \quad (4) \]

Multiplying the left side of (4) by $zd(x)$, we have
\[ zd(x)d(x)y[d(x), z] = 0, \quad \forall \ x, y, z \in R. \quad (5) \]

Again, multiplying the left side of (4) by $d(x)z$, we have
\[ d(x)zd(x)y[d(x), z] = 0, \quad \forall \ x, y, z \in R. \quad (6) \]

Subtracting (6) from (5), we have $[d(x), z]d(x)y[d(x), z] = 0$, Hence
\[ [d(x), z]R[d(x), z] = \{0\} \]

for all $x, z \in R$. Since $R$ is semiprime, we have $[d(x), z] = 0$ for all $x, z \in R$. Therefore, $d$ is a mapping from $R$ into $Z(R)$.

**Theorem 3.2** Let $R$ be a semiprime $*$-ring. If $T : R \to R$ is an additive mapping such that $T(xy) = T(x)y^*$ for all $x, y \in R$, then $T$ maps from $R$ to $Z(R)$.

**Proof.** By hypothesis, we have
\[ T(xy) = T(x)y^*, \quad \forall \ x, y \in R. \quad (7) \]

Now
\[ T(xyz) = T(x(zy)) = T(x)(zy)^* = T(x)y^*z^*, \quad \forall \ x, y, z \in R. \quad (8) \]

On the other hand, we have
\[ T(xzy) = T((xz)y) = T(xz)y^* = T(x)z^*y^*, \quad \forall \ x, y, z \in R. \quad (9) \]

Combining (8) with (9), we get
\[ T(x)[z^*, y^*] = 0, \quad \forall \ x, y, z \in R. \quad (10) \]
Replacing $z$ by $z^*$ and $y$ by $y^*$ in (10), we have

$$T(x)[z, y] = 0, \quad \forall \ x, y, z \in R. \quad (11)$$

Taking $zT(x)$ instead of $z$ in (11), we have

$$T(x)z[T(x), y] = 0, \quad \forall \ x, y, z \in R. \quad (12)$$

Multiplying the left side by $yT(x)$ in (12), we obtain

$$yT(x)T(x)z[T(x), y] = 0, \quad \forall \ x, y, z \in R. \quad (13)$$

Multiplying the left side by $T(x)y$ in (12), we obtain

$$T(x)yT(x)z[T(x), y] = 0, \quad \forall \ x, y, z \in R. \quad (14)$$

Subtracting (14) from (13), we have $[T(x), y]T(x)z[T(x), y] = 0$, which implies that $[T(x), y]R[T(x), y] = \{0\}$ for all $x, y \in R$. Since $R$ is semiprime, we have $[T(x), y] = 0$ for all $x, y \in R$. Therefore, $T$ is a mapping from $R$ into $Z(R)$.

**Theorem 3.3** Let $R$ be a prime $*$-ring. If $R$ admits an $*$-derivation $d$ of $R$ such that $d(x) \neq x$ and $d(xy) = d(x)d(y)$ for all $x, y \in R$, then $d = 0$.

**Proof.** By hypothesis, we have

$$d(xy) = d(x)y^* + xd(y) = d(x)d(y), \quad \forall \ x, y \in R. \quad (15)$$

Replacing $x$ by $xz$ in (15), we have

$$d(x)d(z)y^* + xzd(y) = d(x)d(z)d(y) = d(x)d(zy) = d(x)(d(z)y^* + zd(y)),$$

which implies that $(x - d(x))zd(y) = 0$ for all $x, y, z \in R$. Hence we have $(x - d(x))Rd(y) = \{0\}$ for all $x, y \in R$. Since $R$ is prime, we have $x - d(x) = 0$ or $d(y) = 0$ for all $x, y \in R$. But $d(x) \neq x$, and so $d(y) = 0$ for all $y \in R$, that is, $d = 0$.

**Theorem 3.4** Let $R$ be a prime $*$-ring. If $R$ admits a $*$-derivation $d$ of $R$ such that $d(x) \neq x^*$ for all $x \in R$ and $d(xy) = d(y)d(x)$ for all $x, y \in R$, then $d = 0$.

**Proof.** By hypothesis, we have

$$d(xy) = d(x)y^* + xd(y) = d(y)d(x), \quad \forall \ x, y \in R. \quad (16)$$

Replacing $y$ by $xy$ in (16), we have

$$d(xy)x^* + xd(y)d(x) = d(xy)d(x) = (d(xy)y^* + xd(y))d(x),$$

which implies that $d(xy)^* - d(x))^* = 0$ for all $x, y \in R$. Hence we have $d(xy)^* - d(x))^* = \{0\}$ for all $x \in R$. Since $R$ is prime, we have $x^* - d(x) = 0$ or $d(x) = 0$ for all $x \in R$. But $d(x) \neq x^*$, and so $d(x) = 0$ for all $x \in R$, that is, $d = 0$. 
Theorem 3.5 Let $R$ be a prime $*$-ring and $a \in R$. If $R$ admits an $*$-derivation $d$ of $R$ and $[d(x), a] = 0$, then $d(a) = 0$ or $a \in Z(R)$.

Proof. By hypothesis, we have

$$[d(xy), a] = 0, \ \forall \ x, y \in R,$$

which implies that $[d(x)y^* + xd(y), a] = 0$ for all $x, y \in R$. That is,

$$d(x)[y^*, a] + [x, a]d(y) = 0, \ \forall \ x, y^* \in R. \quad (18)$$

Replacing $x$ by $a$ in (18), we have $d(a)[y^*, a] = 0$ for all $y \in R$. Substituting $y^*$ for $y$ in this relation, we have $d(a)[y, a] = 0$ for all $y \in R$. Again, taking $yx$ instead of $y$ in the last relation, we obtain

$$d(a)y[x, a] = 0, \ \forall \ x, y \in R. \quad (19)$$

This implies that $d(a)R[x, a] = \{0\}$ for all $x \in R$. Since $R$ is prime, we have $d(a) = 0$ or $a \in Z(R)$.

Theorem 3.6 Let $R$ be a semiprime $*$-ring. If $R$ admits an reverse $*$-derivation $d$ of $R$, then $[d(x), z] = 0$ for all $x, z \in R$.

Proof. By hypothesis, we have

$$d(xy) = d(y)x^* + yd(x), \ \forall \ x, y \in R. \quad (20)$$

Replacing $x$ by $xz$ in (20), we have

$$d((xz)y) = d(y)(xz)^* + yd(xz)$$

$$= d(y)z^*x^* + y(d(z)x^* + zd(x))$$

$$= d(y)z^*x^* + yd(z)x^* + yzd(x) \quad (21)$$

for every $x, y, z \in R$. On the other hand, we have

$$d(x(zy)) = d(zy)x^* + zyd(x)$$

$$= (d(y)z^* + yd(z))x^* + zyd(x)$$

$$= d(y)z^*x^* + yd(z)x^* + zyd(x) \quad (22)$$

for every $x, y, z \in R$. Comparing (21) and (22), we get $[y, z]d(x) = 0$ for all $x, y, z \in R$. Substituting $d(x)y$ for $y$ in this relation, we obtain

$$[d(x), z]yd(x) = 0, \ \forall \ x, y, z \in R. \quad (23)$$
Multiplying the right side of (23) by \( zd(x) \), we have
\[
[d(x), z]yd(x)zd(x) = 0, \quad \forall \ x, y, z \in R.
\] (24)

Multiplying the right side of (23) by \( d(x)z \), we have
\[
[d(x), z]yd(x)d(x)z = 0, \quad \forall \ x, y, z \in R.
\] (25)

Subtracting (25) from (24), we have
\[
[d(x), z]yd(x)d(x)z = 0, \quad \forall \ x, y, z \in R.
\]

This implies that \([d(x), z] = 0\) for all \( x, z \in R \).

Theorem 3.7 Let \( R \) be a prime \(*\)-ring. If \( R \) admits an \(*\)-derivation \( d \) of \( R \) such that \( d([x, y]) = 0 \) for all \( x, y \in R \), then \( d = 0 \) or \( R \) is commutative.

Proof. By hypothesis, we have
\[
d([x, y]) = 0, \quad \forall \ x, y \in R.
\] (26)

Replacing \( x \) by \( xy \) in (26), we have
\[
d([x, y]y) = d([x, y])y^* + [x, y]d(y) = 0
\] for all \( x, y \in R \). By the relation (26), we have \([x, y]d(y) = 0\) for all \( x, y \in R \).

Substituting \( sx \) for \( x \) in this relation, we have \([s, y]d(y) = 0\) for all \( s, y \in R \). This implies that \([s, y] = 0\) for all \( s, y \in R \).

Theorem 3.8 Let \( R \) be a prime \(*\)-ring. If \( R \) admits an \(*\)-derivation \( d \) of \( R \) such that \( d(x \circ y) = 0 \) for all \( x, y \in R \), then \( d = 0 \) or \( R \) is commutative.

Proof. By hypothesis, we have
\[
d(x \circ y) = 0, \quad \forall \ x, y \in R.
\] (27)

Replacing \( x \) by \( xy \) in (27), we have
\[
d((x \circ y)y) = d(x \circ y)y^* + (x \circ y)d(y) = 0
\] for all \( x, y \in R \). By the relation (27), we have \((x \circ y)d(y) = 0\) for all \( x, y \in R \).

Substituting \( sy \) for \( x \) in this relation, we have \((s \circ y)yd(y) = 0\) for all \( y, s \in R \).

This implies that \((s \circ y)Rd(y) = \{0\}\) for all \( s, y \in R \). Since \( R \) is prime, we
have \((s \circ y) = 0\) or \(d(y) = 0\) for all \(s, y \in R\). Let \(K = \{y \in R | d(y) = 0\}\) and 
\(L = \{y \in R | s \circ y = 0, \forall s \in R\}\). Then \(K\) and \(L\) are both additive subgroups and \(K \cup L = R\), but \((R, +)\) is not union of two its proper subgroups, which implies that either \(K = R\) or \(L = R\). In the former case, we have \(d = 0\). On the other hand, if \(L = R\), then we have \(s \circ y = 0\) for all \(s, y \in R\). Replacing \(s\) by \(sz\) in the last relation, we obtain \(s[z, y] = 0\) for all \(s, y, z \in R\). That is, \(R[z, y] = \{0\}\). This implies that \(xR[z, y] = \{0\}\) for \(0 \neq x \in R\). Since \(R\) is prime, we have \([z, y] = \{0\}\) for all \(y, z \in R\), which means that \(R\) is commutative.

**Theorem 3.9** Let \(R\) be a prime \(*\)-ring. If \(R\) admits an \(*\)-derivation \(d\) of \(R\) such that \(d(x) \circ y = 0\) for all \(x, y \in R\), then \(d = 0\) or \(R\) is commutative.

**Proof.** By hypothesis, we have
\[
d(x) \circ y = 0, \forall x, y \in R.
\]
Replacing \(x\) by \(xz\) in (28), we have
\[
(d(x) \circ y)z^* + d(x)[z^*, y] + x(d(z) \circ y) - [x, y]d(z) = 0
\]
for all \(x, y, z \in R\). By using the relation (28), we obtain \(d(x)[z^*, y] - [x, y]d(z) = 0\) for all \(x, y, z \in R\). Substituting \(x\) for \(y\) in this relation, we get \(d(x)[z^*, x] = 0\) for all \(x, z \in R\). Again, replacing \(z\) by \(z^*\) in the last relation, we have \(d(x)[z, x] = 0\) for all \(x, y \in R\).

Using the arguments of the last part in proof of Theorem 3.7, we get the required result.

**Theorem 3.10** Let \(R\) be a prime \(*\)-ring. If \(R\) admits an \(*\)-derivation \(d\) of \(R\) such that \([d(x), y] = [x, y]\) for all \(x, y \in R\), then \(d = 0\) or \(R\) is commutative.

**Proof.** By hypothesis, we have
\[
[d(x), y] = [x, y], \forall x, y \in R.
\]
Replacing \(x\) by \(xz\) in (29), we have \([d(xz), y] = [xz, y]\) for all \(x, y, z \in R\), which implies that \([d(x)z^*, y] + [zd(z), y] = x[z, y] + [x, y]z\) for all \(x, y, z \in R\). That is, \(d(x)[z^*, y] + [d(x), y]z^* + x[d(z), y] + [x, y]d(z) = x[z, y] + [x, y]z\) for all \(x, y, z \in R\). Substituting \(y\) for \(x\) in the above relation and using (29), we have
\[
d(x)[z^*, y] = 0 \forall x, y, z \in R.
\]
Again, replacing \(y\) by \(yx\) in (30), we have \(d(x)y[z^*, x] = 0\) for all \(x, y, z \in R\). Hence \(d(x)R[z^*, x] = \{0\}\) for all \(x, z \in R\). Since \(R\) is prime, we have \(d(x) = 0\) or \([z^*, x] = 0\) for all \(x, z \in R\).

Using the arguments of the last part in proof of Theorem 3.7, we get the required result.
References


Received: February 9, 2017; Published: March 16, 2017