A Note on $UU$ Rings

Peter V. Danchev

Mathematical Department, Plovdiv University, Bulgaria

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Abstract

A give a new simpler proof of the following result due to Danchev-Lam (Publ. Math. Debrecen, 2016): A ring $R$ has only unipotent units if, and only if, 2 is a nilpotent in $R$ and the unit group $U(R)$ is a 2-group.

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1. Introduction

All our rings $R$ are associative and unital as their unit groups are denoted by $U(R)$.

Mimicking Călugăreanu, one states the following notion:

Definition 1.1. A ring is called $UU$ if each its unit is unipotent, that is, the sum of 1 and a nilpotent.

In [3] was given a systematic exploration of these rings and, resulting, a necessary and sufficient condition was established when an arbitrary ring is a $UU$ ring. The purpose of this short note is to demonstrate a new proof of the aforementioned criterion. This will be done in the upcoming section.

2. The Theorem

Our method of proof is based on two ideas from [1] and [2], thus considerably simplifying the proof given in [3]. However, it is stated there in a more conceptual form. And so, before doing that, we need the next simple but useful tools:
Fact 1. ([1, Proposition 2.29]) A ring $R$ of char($R$) = 2 is a UU ring if and only if $U(R)$ is a 2-group.

Fact 2. ([3, Theorem 2.4(1)]) For any nil-ideal $I \triangleleft R$, the ring $R$ is UU if, and only if, the factor-ring $R/I$ is UU.

Fact 3. (Folklore) For any nil-ideal $I$ of the ring $R$ the next group isomorphism holds:

$$U(R)/(1+I) \cong U(R/I).$$

Indeed, since $I$ is nil and so $1+I \leq U(R)$, the natural map $\varphi : R \to R/I$ can be restricted to the surjective homomorphism $\varphi_{[U(R) : U(R) \to U(R/I)}$ with kernel $1+I$. We further just need to apply the classical Homomorphism's Theorem to get the desired relation after all.

We are now ready to proceed by proving our main statement.

Theorem 2.1. A ring $R$ is UU if, and only if, char($R$) is a power of 2 and $U(R)$ is a 2-group.

Proof. "Necessity". Since by definition $-1 \in 1 + \text{Nil}(R)$, it must be that $2 \in \text{Nil}(R)$, so that $2^t = 0$ in $R$ and thus char($R$) = $2^t$ for some $t \in \mathbb{N}$. Setting $I = 2R$, one sees that $I$ is a nil-ideal of $R$ with $I^t = 2^{t-1}I = \{0\}$. In view of Fact 2 it follows that $R/I$ is a UU ring of characteristic 2 and this, accomplished with Fact 1, both assure that $U(R/I)$ is a 2-group. However, we claim that $1+I$ is a bounded 2-group (compare with [2, Theorem 2.3]). In fact, $(1+I)^{2^t} = 1 + I^{2^t} = 1 + (2R)^{2^t} = \{1\}$. This follows because for any $z \in R$ we write that $(1+2z)^{2^t} = \sum_{j=0}^{2^t} C_j^{2^t} 2^j z^j = 1$, where $C_j^{2^t}$ is the binomial coefficient of $2^t$ over $j$, and also calculate that the integer $C_j^{2^t} 2^j$ is divisible by the integer $2^t$ whenever $t \geq 0$ and $1 \leq j \leq t$. To verify the last assertion, for an arbitrary binomial coefficient $C_i$ it is true the recurrent equality $C_i = \frac{n}{i} C_{i-1}^{n-1}$, whence one writes that $C_j^{2^t} 2^j = 2^t \frac{j}{j} C_{j-1}^{2^t-1}$. Furthermore, suppose that $j = 2^k m$, where $(2, m) = 1$ and $0 \leq k < j$ because $j \geq 2^k > k$. Consequently, $C_j^{2^t} 2^j = 2^t \frac{2^t-1}{m} C_{j-1}^{2^t-1}$. But $m$ cannot be canceled by any power of 2, and hence $C_{j-1}^{2^t-1}$ must be an integer, as required. This substantiates our claim.

We therefore appeal to Fact 3, namely that $U(R/I) \cong U(R)/(1+I)$, and by what we have just already shown so far one can conclude that $U(R)$ is also a 2-group, as pursued.

"Sufficiency". Again put $I = 2R$ and observe that it is a nil-ideal of $R$ because 2 is a nilpotent in $R$. Since by supposition $U(R)$ is a 2-group, any its factor-group is again a 2-group and thus Fact 3 is applicable to infer that so is $U(R/I)$. That is why, Fact 1 ensures that the quotient $R/I$ is a UU ring, and so Fact 2 guarantees that the same holds for $R$, sustaining our assertion. □
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References


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